CHAPTER 10

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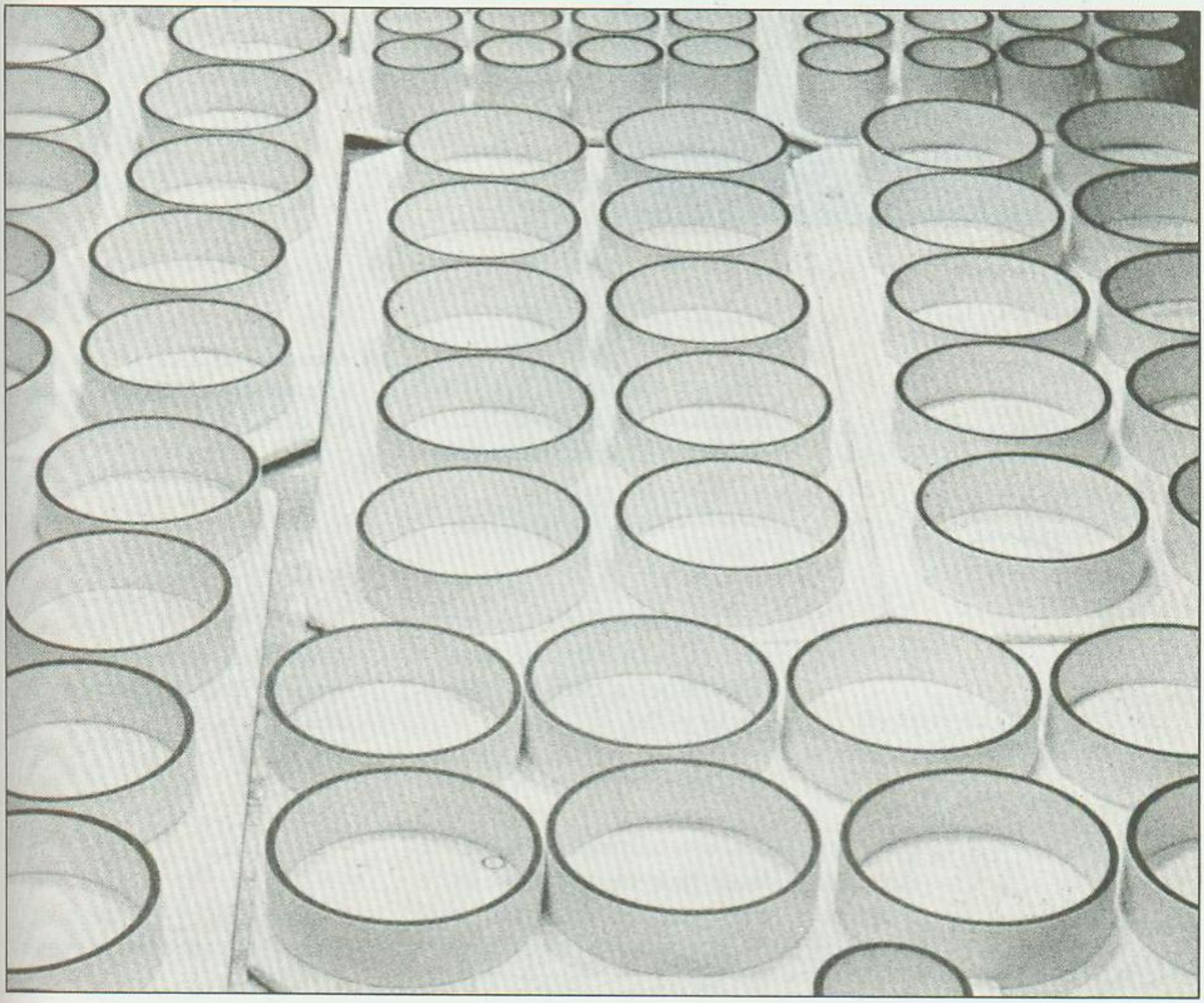
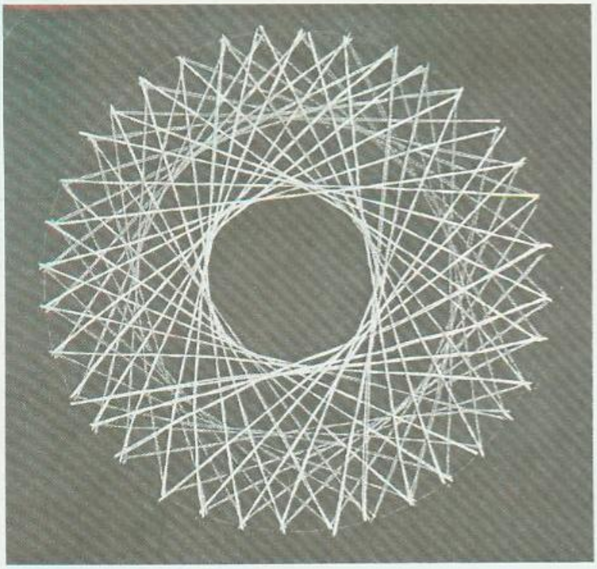
Circles

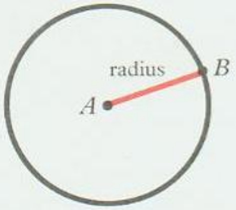


10–1 Basic Definitions



Recall that a circle is a set of points in a plane that are all an equal distance from a fixed point.

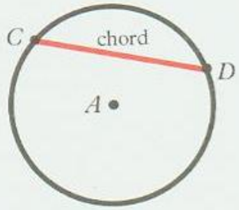
In this lesson we shall define terms related to circles.



$\overline{AB}$ is a *radius* of $\odot A$. Each point on the circle is the endpoint of another radius.

Definition 10–1

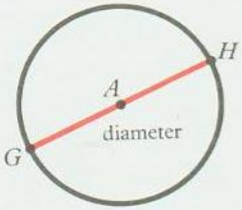
A **radius** of a circle is a segment whose endpoints are the center and a point on the circle.



$\overline{CD}$ is a *chord* of $\odot A$. Each pair of points on the circle determines a chord of the circle.

Definition 10–2

A **chord** of a circle is a segment with endpoints on the circle.

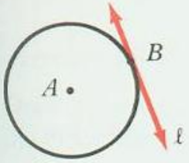


$\overline{GH}$ is a *diameter* of $\odot A$. Each pair of points on the circle that are collinear with A determine a diameter of the circle.

Definition 10–3

A **diameter** of a circle is a chord that contains the center of the circle.

Chords, radii, and diameters are segments associated with circles. The following definitions describe some lines and angles that are also associated with circles.



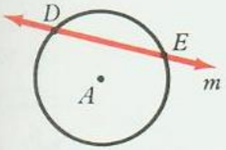
Line ℓ has only point B in common with $\odot A$.

Line ℓ is *tangent* to $\odot A$.

Point B is called the *point of tangency*.

Definition 10-4

A **tangent** to a circle is a line that intersects the circle in exactly one point.

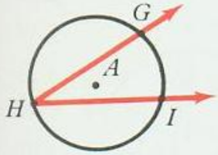


Line m has two points in common with $\odot A$.

Line m is a *secant* of $\odot A$.

Definition 10-5

A **secant** of a circle is a line that intersects the circle in exactly two points.

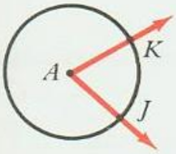


The vertex of $\angle GHI$ is on $\odot A$. The sides of $\angle GHI$ intersect $\odot A$ in points G and I.

$\angle GHI$ is an *inscribed angle*.

Definition 10-6

An **inscribed angle** is an angle with vertex on a circle and with sides that contain chords of the circle.

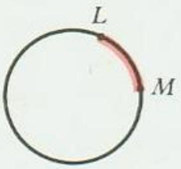


The vertex of $\angle KAJ$ is the center of $\odot A$.

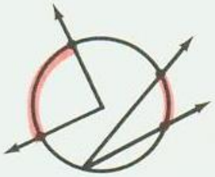
$\angle KAJ$ is a *central angle*.

Definition 10-7

A **central angle** is an angle with vertex at the center of a circle.

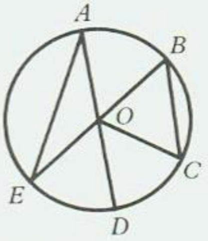


We think of an *arc* as a continuous part of a circle. We write $\widehat{LM}$ to symbolize *arc LM*. An *intercepted arc* is an arc with endpoints on the sides of inscribed or central angles.



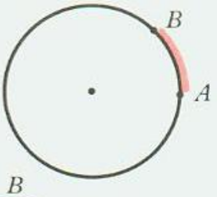
EXERCISES

A.

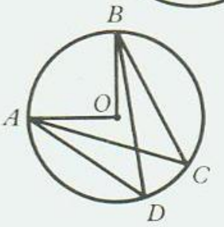


(Exs. 1-4)

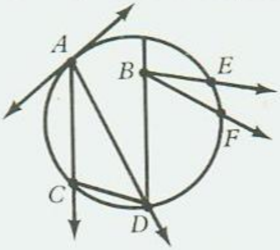
1. Name all the chords shown in this figure.
2. Name the diameters in the figure.
3. Name at least four arcs in the figure.
4. Name all the radii shown.
5. Draw a circle with center T. Draw a line tangent to the circle at point B. Draw a secant line that intersects the circle in points F and G.
6. Draw a circle and an inscribed angle. If the arc intercepted by that angle is $\overset{\frown}{PQ}$, label it on the circle.
7. Draw a circle and a central angle. If the arc intercepted by that angle is $\overset{\frown}{XY}$, label it on the circle.
8. Draw a circle and label an arc as shown. Then draw an inscribed angle that intercepts $\overset{\frown}{AB}$.



9. Draw a circle and label an arc $\overset{\frown}{CD}$. Draw two different inscribed angles, both with intercepted arc $\overset{\frown}{CD}$.
10. Name all angles with intercepted arc AB; with intercepted arc CD.

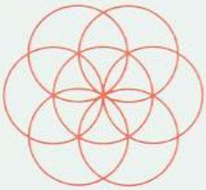


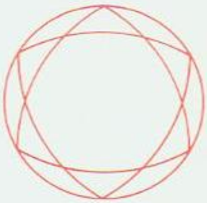
11. Name all the inscribed angles.

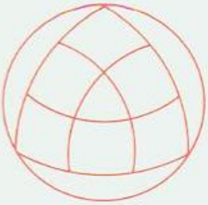


Activity

Use your compass to make larger examples of each *compass art* design below. Make some other designs of your own and color them in an interesting way.

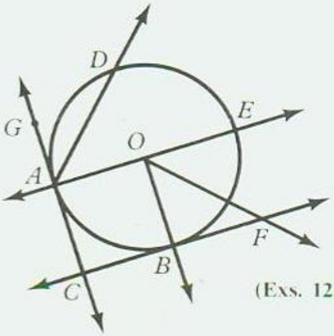






12. Name all angles with at least one side on a tangent line.

13. Name all the central angles.



(Exs. 12, 13)

B.

For exercises 14–16 draw a circle A.

14. Draw a segment that has one endpoint on the circle but is not a chord.

15. Draw a segment that is completely inside the circle but is not a chord.

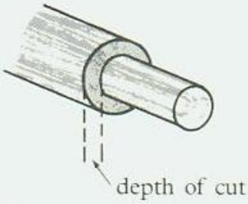
16. Draw a segment that intersects the circle in two points, contains the center, but is not a radius, diameter, or chord.

For exercises 17–19 draw a circle B.

17. Draw a line that is neither a secant line or a tangent line.

18. Draw an angle that has its vertex on the circle and intersects the circle but is not an inscribed angle.

19. Draw an angle with two sides intersecting the circle and vertex not outside the circle, but which is not a central angle or an inscribed angle.



20. A handle 0.41 cm in diameter is reduced to a diameter of 0.34 cm. What is the depth of the cut?

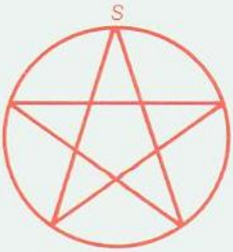
21. Planes that fly from New York to Paris often fly over Ireland. Why is this route chosen? (*Hint:* Use a globe and a flexible ruler.)

PROBLEM SOLVING

This *star polygon* can be made by

a. marking five evenly spaced points on a circle. (You may want to use a protractor.)

b. starting at point S and connecting every second point as you go in a clockwise direction around the circle.

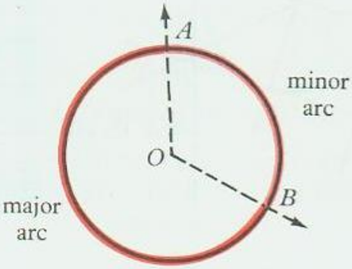


It is called star polygon $\left\{\begin{matrix}5\\2\end{matrix}\right\}$.

1. Can you make star polygons $\left\{\begin{matrix}7\\2\end{matrix}\right\}$, $\left\{\begin{matrix}7\\3\end{matrix}\right\}$, $\left\{\begin{matrix}8\\3\end{matrix}\right\}$, $\left\{\begin{matrix}9\\2\end{matrix}\right\}$, and $\left\{\begin{matrix}9\\4\end{matrix}\right\}$?

2. What can you say about star polygons $\left\{\begin{matrix}7\\5\end{matrix}\right\}$, $\left\{\begin{matrix}7\\4\end{matrix}\right\}$, $\left\{\begin{matrix}8\\5\end{matrix}\right\}$, etc.?

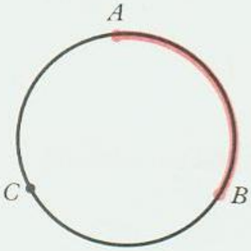
10–2 The Degree Measure of Arcs



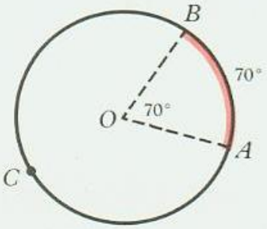
When two points (not endpoints of a diameter) are selected on a circle, two arcs are determined. One is called the *major arc,* the other the *minor arc.*

Definition 10-8

A **minor arc** is an arc that lies in the interior of a central angle. Otherwise it is called a **major arc.**



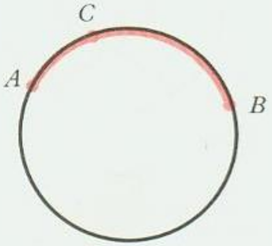
$\widehat{AB}$ will always denote the minor arc determined by points A and B. To denote the major arc a third point is included. $\widehat{ACB}$ denotes the major arc determined by points A and B.



The measure of an arc is determined by the measure of its central angle. For example, $m\widehat{AB} = m\angle AOB = 70$, and $m\widehat{ACB} = 360 - 70 = 290$.

Definition 10-9

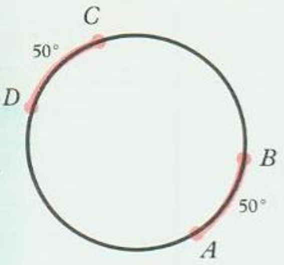
The **measure of a minor arc** is the measure of its associated central angle. The **measure of a major arc** is 360 minus the measure of its associated minor arc.



Point C is on arc AB. Two arcs, $\widehat{AC}$ and $\widehat{CB}$, are added to give arc AB.

Arc Addition Postulate

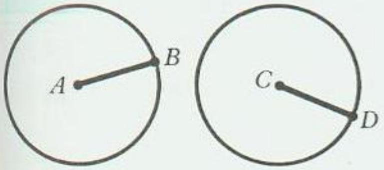
If C is on $\widehat{AB}$, then $m\widehat{AC} + m\widehat{CB} = m\widehat{AB}$.



Arc DC and arc BA on this circle both have measure $50°$. We say that these *arcs are congruent*.

Definition 10–10

If two **arcs** of a circle have the same measure, they are called **congruent.** If $\widehat{AB}$ and $\widehat{CD}$ are congruent, we write $\widehat{AB} \cong \widehat{CD}$.



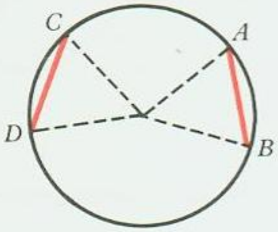
Radius AB and radius CD are the same length. The circles they determine are congruent.

Definition 10–11

Two **circles** are **congruent** if they have radii of equal length.

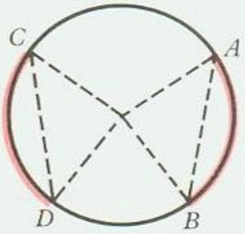
These two figures should focus your attention on the relationship between congruent chords and their arcs.

Given congruent chords
$\overline{AB} \cong \overline{CD}$.



Is $\widehat{AB} \cong \widehat{CD}$?
Why?

Given congruent arcs
$\widehat{AB} \cong \widehat{CD}$.



Is $\overline{AB} \cong \overline{CD}$?
Why?

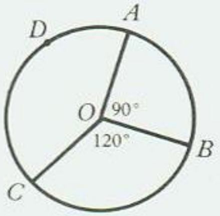
These figures suggest the following theorems.

Theorem 10–1 In a circle or in congruent circles congruent chords have congruent minor arcs.

Theorem 10–2 In a circle or in congruent circles congruent minor arcs have congruent chords.

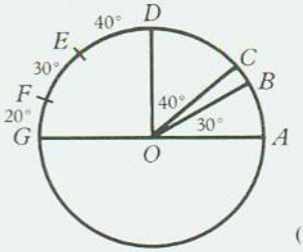
EXERCISES

A.



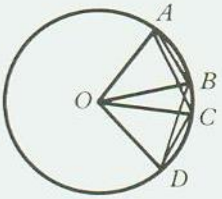
(Exs. 1-3)

1. Is point D on $\widehat{BAC}$? Is D on $\widehat{AB}$?
2. Name three minor arcs. Name three major arcs. Name three central angles.
3. Find each of these degree measures.
   a. $m\widehat{AB}$ b. $m\widehat{AC}$
   c. $m\widehat{BCA}$ d. $m\widehat{ABC}$



(Exs. 4-8)

4. Find each of these degree measures.
   a. $m\widehat{AB}$ b. $m\angle BOC$
   c. $m\widehat{BD}$ d. $m\angle COG$
5. Find each of these degree measures.
   a. $m\widehat{DF}$ b. $m\widehat{EG}$ c. $m\widehat{ECG}$
6. Name all the minor arcs containing point C.
7. Name three pairs of congruent minor arcs.
8. Name a pair of congruent major arcs.

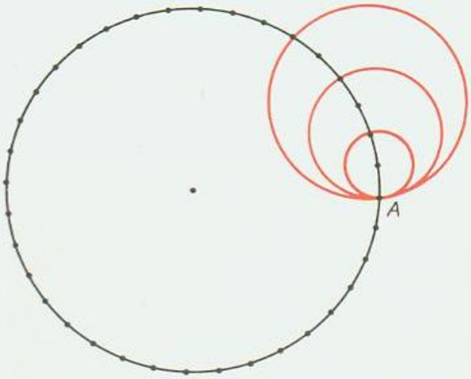


(Ex. 9)

9. Suppose $m\angle AOB = 40$, $m\angle BOC = 20$, and $m\angle COD = 40$. Name all pairs of congruent chords.

Activity

1. Draw a circle with a radius of 4 cm and label a point A on the circle.
2. Using a protractor, draw points each 10° around the circle, beginning with point A.
3. Each of these points is the center of a circle through point A. Draw these circles. (Three such circles have been drawn.)
4. Guess what the final figure looks like.



B.

For exercises 10–12, make accurate drawings and answer the questions.

10. In a circle, is the chord of a 90° arc twice as long as the chord of a 45° arc?

11. If the measure of a minor arc is doubled, is the measure of the central angle doubled?

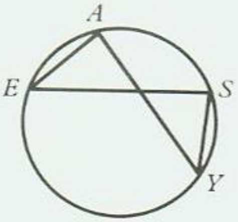
12. Suppose in circle O, $\overline{OA}$ and $\overline{OB}$ are radii and $OA = OB = AB$. What is the measure of $\angle AOB$?

13. Prove Theorem 10–1.

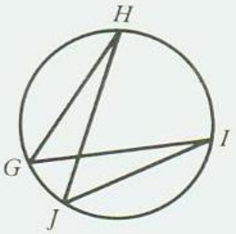
14. Prove Theorem 10–2.

C.

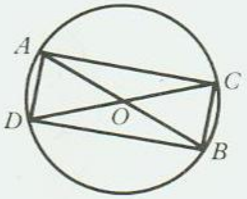
15. Given: $ES = AY$
Prove: $EA = SY$



16. Given: $\widehat{HG} \cong \widehat{JI}$
Prove: $HJ = GI$

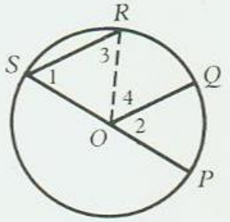


17. Given: $\overline{AB}$ and $\overline{CD}$ are diameters.
Prove: $ACBD$ is a parallelogram.



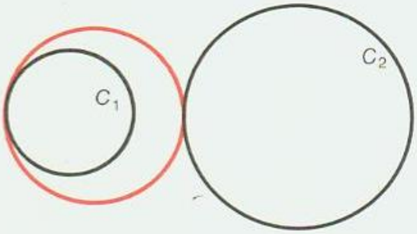
18. Prove that if the vertices of an equilateral triangle are on a circle, the circle is divided into three congruent arcs.

19. Given: $\overline{POS}$ is a diameter of $\odot O$.
$\overline{SR} \parallel \overline{OQ}$
Prove: $\widehat{RQ} \cong \widehat{QP}$

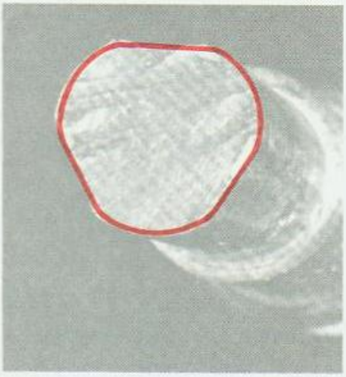


PROBLEM SOLVING

1. Draw circles C_1 and C_2 of two different sizes.
2. How many circles with collinear centers are there that touch both of the given circles at exactly one point? (One of them is drawn.)
3. Construct each one of these circles. (First construct the centers of these circles. No guessing is necessary.)



10–3 Chords and Distances from the Center

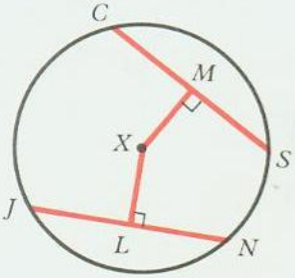


Pictured is an electrician's drill bit whose shank end has three flat surfaces. One necessary feature of this design is that these flat surfaces must be an equal distance from the rotating axis to ensure that the drill runs smoothly.

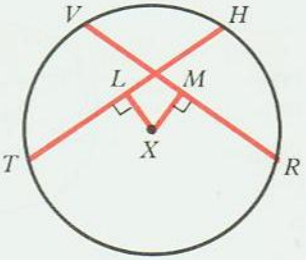
The theorem in this lesson describes a method for determining that this condition is satisfied.

In each figure a pair of congruent chords is given.

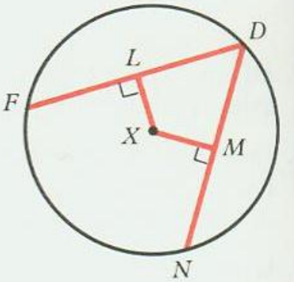
$CS = JN$



$VR = TH$



$FD = DN$



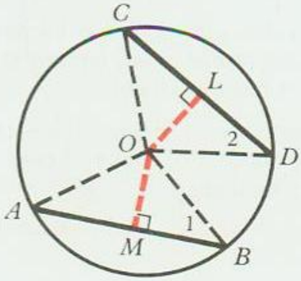
In each case does $XL = XM$?

These examples suggest the following theorem.

Theorem 10–3 In a circle or in congruent circles congruent chords are equidistant from the center.

PROOF

Given: $\odot O$, $\overline{AB} \cong \overline{CD}$, $\overline{OM} \perp \overline{AB}$, $\overline{OL} \perp \overline{CD}$
Prove: $OM = OL$

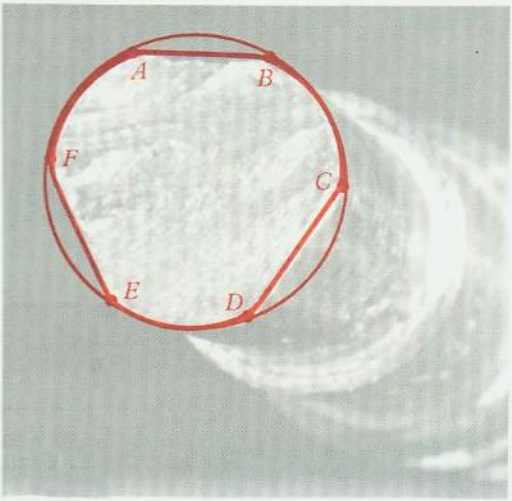


Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	1. Given
2. $OA = OB = OC = OD$	2. Definition of circle
3. $\overline{OA} \cong \overline{OB} \cong \overline{OC} \cong \overline{OD}$	3. Definition of congruent segments
4. $\triangle AOB \cong \triangle COD$	4. SSS Congruence
5. $\angle 1 \cong \angle 2$	5. CPCTC
6. $\overline{OM} \perp \overline{AB}$, $\overline{OL} \perp \overline{CD}$	6. Given
7. $\angle OMB \cong \angle OLD$, $\angle OMB$ and $\angle OLD$ are right angles.	7. Perpendicular lines form congruent right angles.
8. $\triangle OMB$ and $\triangle OLD$ are right triangles.	8. Definition of right triangles
9. $\triangle OMB \cong \triangle OLD$	9. HA Congruence
10. $\overline{OM} \cong \overline{OL}$	10. CPCTC
11. $OM = OL$	11. Definition of congruent segments

APPLICATION

The drill bit shown here must be balanced in order for the drill to run smoothly. How should the shank end be constructed so that the drill bit is balanced?

If the segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ are congruent, then Theorem 10–3 says that they are equidistant from the center of the circle containing arcs $\widehat{AB}$, $\widehat{CD}$, and $\widehat{EF}$. This will ensure that the drill bit is balanced.



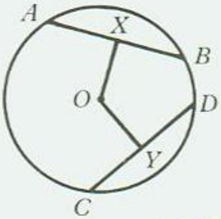
The converse of Theorem 10–3 is also important. It is stated below.

Theorem 10–4 In a circle or in congruent circles chords equidistant from the center are congruent.

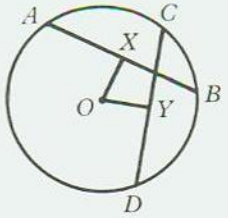
EXERCISES

A.

1. **Given:** $AB = CD$
   $\overline{OX} \perp \overline{AB}$, $\overline{OY} \perp \overline{CD}$
   $OX = 3$
   Find: OY.

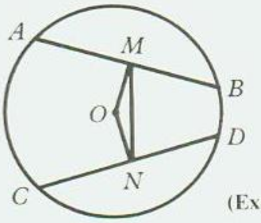


2. **Given:** $OX = OY$
   $\overline{OX} \perp \overline{AB}$, $\overline{OY} \perp \overline{CD}$
   $AB = 10$
   Find: CD



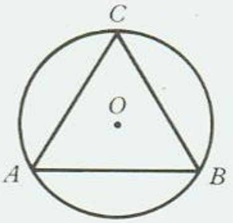
3. **Given:** $\overline{OM} \perp \overline{AB}$, $\overline{ON} \perp \overline{CD}$
   $AB = CD$
   $m\angle MON = 150$
   Find: $m\angle OMN$ (*Hint:* Does $OM = ON$?)

4. **Given:** $\overline{OM} \perp \overline{AB}$, $\overline{ON} \perp \overline{CD}$
   $AB = CD$
   $m\angle NMB = 70$
   Find: $m\angle MON$



(Exs. 3, 4)

5. **Given:** $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are equidistant from the center O.
   Prove: $\triangle ABC$ is equilateral.



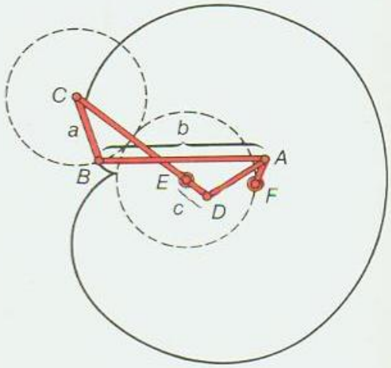
Activity

The path of a point B on a circle as that circle rolls around a fixed circle of the same radius is a curve called a *cardioid*.

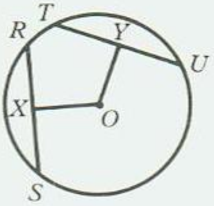
A cardioid can also be drawn using a linkage as shown in the figure.

Follow these directions and draw a cardioid.

1. Construct a linkage out of stiff cardboard like the one shown in the figure. Note that $BC = AD = a$, $AB = CD = b$, $DE = AF = c$, and $a^2 = bc$. (Suggestion: Let $a = 6$ cm, $b = 32$ cm, and $c = 8$ cm.)
2. Attach your linkage to a bulletin board at points E and F.
3. Place your pencil at B and draw.

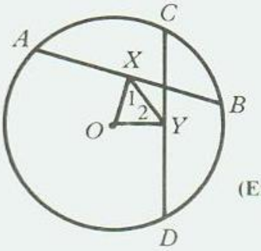


B.



6. Given: $OX = OY$
$\overline{OX} \perp \overline{RS}$, $\overline{OY} \perp \overline{TU}$
Prove: $m\overset{\frown}{RS} = m\overset{\frown}{TU}$

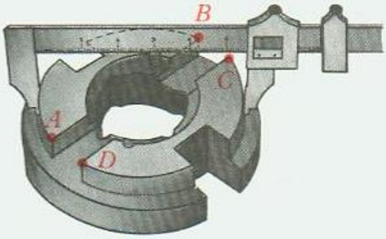
7. Given: In circle O, $\overline{OX} \perp \overline{AB}$,
$\overline{OY} \perp \overline{CD}$
$m\angle 1 = m\angle 2$
Prove: $AB = CD$



(Exs. 7, 8)

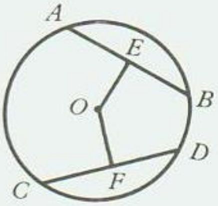
8. Given: In circle O, $\overline{OX} \perp \overline{AB}$,
$\overline{OY} \perp \overline{CD}$
$CD = AB$
Prove: $m\angle 1 = m\angle 2$

9. Often machines parts like the one shown here will work properly only if the groove is "centered." Why does measuring $\overline{AB}$ and $\overline{CD}$ tell you whether or not the groove is centered?

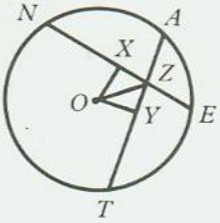


C.

10. This exercise structures a proof of Theorem 10–4.
Given: Circle O, $OE = OF$
$\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$
Prove: $AB = CD$
(*Hint:* Draw radii and use congruent triangles.)

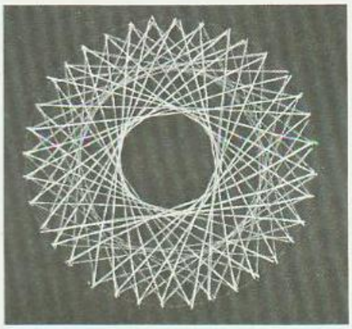


11. Given: In circle O, $\overline{OX} \perp \overline{NE}$,
$\overline{OY} \perp \overline{AT}$
$\angle XZO \cong \angle YZO$
Prove: $NE = AT$



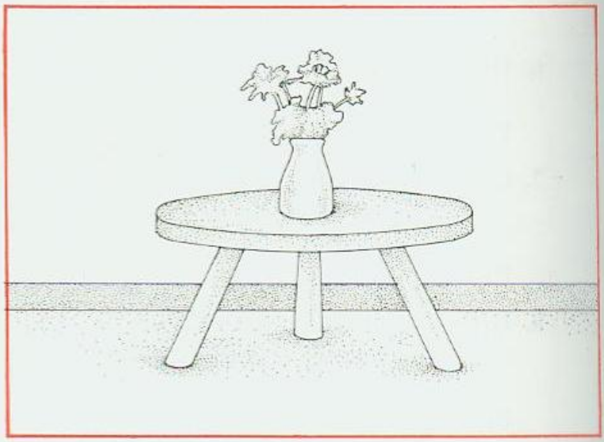
PROBLEM SOLVING

The middle region of this string design appears to be circular. Explain why this is true. That is, explain why the midpoints of those segments that border on the middle region are equidistant from an imaginary center point. (You may assume that the nails at the ends of each piece of string are equally spaced around a circle.)

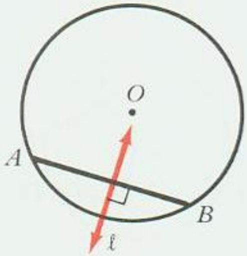


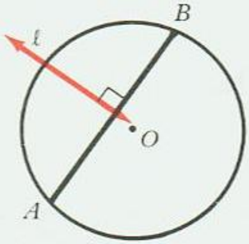
10–4 Perpendiculars to Chords

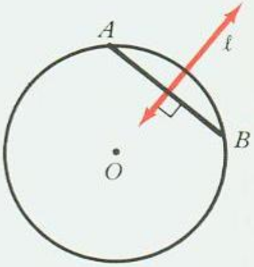
A carpenter's square can be used to find the center of a round table. The theorem in this section explains how this can be done.



These figures illustrate a property of the perpendicular bisector of a chord. In each figure ℓ is the perpendicular bisector of chord $\overline{AB}$.







Does line ℓ pass through point O?

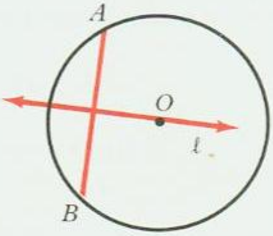
These three figures suggest the following theorem.

Theorem 10–5 The perpendicular bisector of a chord contains the center of the circle.

PROOF

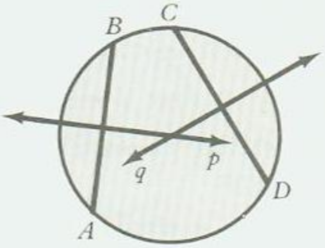
Given: $\overline{AB}$ is a chord of circle O, and ℓ is the perpendicular bisector of $\overline{AB}$.

Prove: O is a point of ℓ.



Statements	Reasons
1. ℓ is the perpendicular bisector of of $\overline{AB}$.	1. Given
2. $OA = OB$	2. Definition of circle
3. O lies on ℓ.	3. A point equidistant from points A and B belongs to the perpendicular bisector of $\overline{AB}$. (Theorem 6-10).

APPLICATION



Find the center of a round table.

Step 1 Select any two chords $\overline{AB}$ and $\overline{CD}$.

Step 2 Draw the perpendicular bisector p of $\overline{AB}$, and the perpendicular bisector q of $\overline{CD}$.

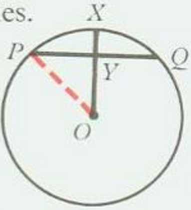
Conclusion: By Theorem 10-5 the center lies on both lines p and q. Consequently, the center of the table must be the intersection of these lines.

Two other important theorems are the following.

Theorem 10-6 If a line through the center of a circle is perpendicular to a chord that is not a diameter, then it bisects the chord and its minor arc.

Theorem 10-7 If a line through the center of a circle bisects a chord that is not a diameter, then it is perpendicular to the chord.

Theorem 10-6 can be used to find missing information about circles.



Example **Given:** $\odot O$ with radius 4 in. $\overline{OX} \perp \overline{PQ}$. Chord $\overline{PQ}$ is 1 inch from O.

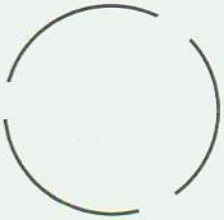
Find: PQ

The given information tells us that $OP = 4$ (why?) and $OY = 1$ (why?). By applying the Pythagorean Theorem to $\triangle OPY$, we can determine that $PY = \sqrt{15}$. Theorem 10-6 tells us that $\overline{OX}$ bisects $\overline{PQ}$. Therefore $PQ = 2\sqrt{15}$.

EXERCISES

A.

1. Trace the figure at the right. Then find the center of the arcs and complete the circle.

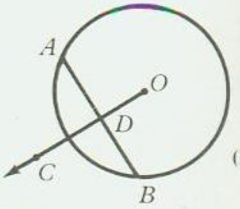


2. Using compass and straightedge, draw a figure that illustrates Theorems 10-6 and 10-7.

In the figure for exercises 3-4, O is the center of the circle.

3. If $\overrightarrow{OC} \perp \overline{AB}$, how is $m\widehat{AC}$ related to $m\widehat{CB}$?

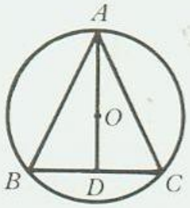
4. If $AD = 3$ and $BD = 3$, what is $m\angle BDC$?



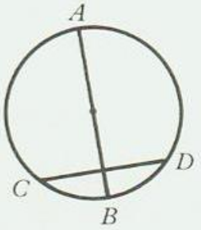
(Exs. 3, 4)

In exercises 5-7, state whether Theorem 10-5, 10-6, or 10-7 is used to reach the conclusion.

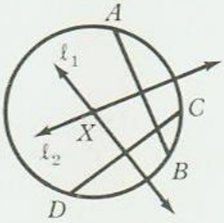
5. **Given:** $\odot O$ with $\overline{AOD}$ an altitude of $\triangle ABC$
Conclusion: $\overline{AD}$ bisects $\overline{BC}$.



6. **Given:** Diameter $\overline{AB}$ bisects $\overline{CD}$.
Conclusion: $\overline{AB} \perp \overline{CD}$

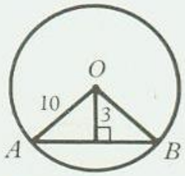


7. **Given:** ℓ_1 is the perpendicular bisector of $\overline{CD}$.
ℓ_2 is the perpendicular bisector of $\overline{AB}$.
ℓ_1 and ℓ_2 intersect at X.
Conclusion: X is the center of the circle.



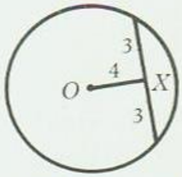
In exercises 8-10, find the missing information. O is the center of each circle.

8.



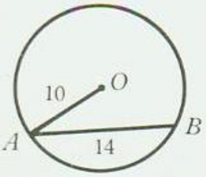
$AB = ?$

9.



What is the length of the radius?

10.

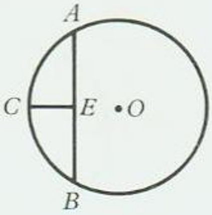


How far from the center is chord $\overline{AB}$?

B.

11. In a circle with radius 5 cm, $\overline{AB}$ is a chord that has length 8 cm. How far is $\overline{AB}$ from the center of the circle? (Draw a sketch to help you.)

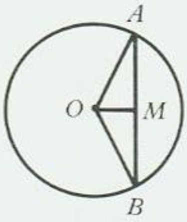
12. **Given:** C is the midpoint of $\overset{\frown}{AB}$.
$\overline{CE} \perp \overline{AB}$
$CE = 2$ inches, $AB = 16$ inches
Find: The length of a radius of the circle



13. **Given:** $AM = MB = 6$ cm
$AO = 10$ cm
Find: OM.

14. **Given:** $\overline{OM} \perp \overline{AB}$
$OM = 5, AO = 13$
Find: AB.

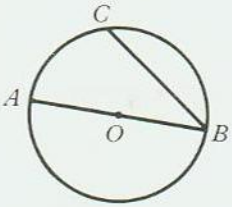
15. **Given:** M is the midpoint of $\overline{AB}$.
$OM = MB$
$OB = \sqrt{2}$
Find: AB



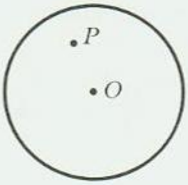
(Exs. 13–15)

16. Two chords in a circle are equal in length. Their distances from the center can be represented by x^2 and $4x$ respectively. How far is each chord from the center?

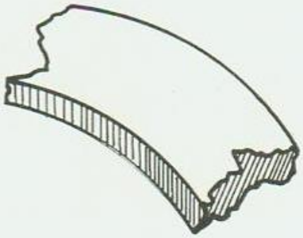
17. **Given:** A circle with $AB = 8$ ft
$m\angle ABC = 45$
Find: The distance from point O to $\overline{BC}$



18. Draw a circle O and mark a point P inside the circle. Using your compass and straightedge, construct a chord that is bisected by P.

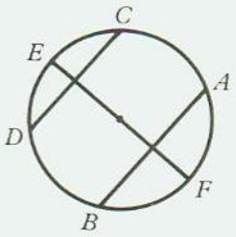


19. A part of an old wheel is found by an archeologist in a dig. The wheel can be reconstructed by determining the original radius. Explain how this can be accomplished.

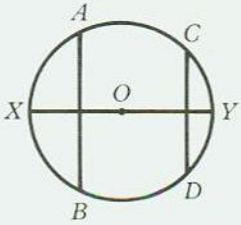


C.

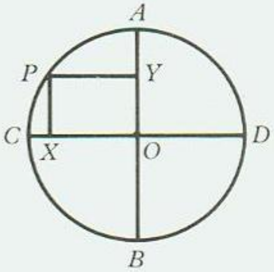
20. **Given:** $\overline{AB} \parallel \overline{CD}$
$\overline{EF}$ is the perpendicular bisector of $\overline{AB}$.
Prove: $\overline{EF}$ bisects $\overline{CD}$.



21. **Given:** $\overline{XY}$ is a diameter.
$\overline{XY}$ bisects $\overline{AB}$.
$\overline{XY}$ bisects $\overline{CD}$.
Prove: $\overline{AB} \parallel \overline{CD}$



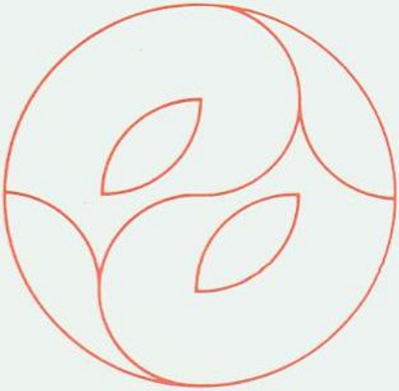
22. **Given:** $\overline{AB}$ and $\overline{CD}$ are diameters and $\overline{AB} \perp \overline{CD}$.
$AO = 10$ cm
$\overline{PX} \perp \overline{CD}$ and $\overline{PY} \perp \overline{AB}$
Find: XY



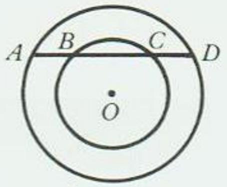
23. A circle O has a radius of 10 cm. Chords $\overline{AB}$ and $\overline{CD}$ are perpendicular and intersect at a point F inside the circle. If $AB = 16$, and $CD = 18$, find DF.

Activity

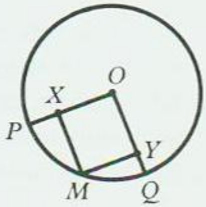
1. Trace this figure and cut out its pieces.
2. Arrange the pieces to form two egg-shaped regions with a missing center.
3. Draw this puzzle with a compass. (You will need to carefully determine the center of each arc. No guessing is necessary.)



24. Given: Two circles with center O
A, B, C, and D are collinear.
Prove: $AB = CD$

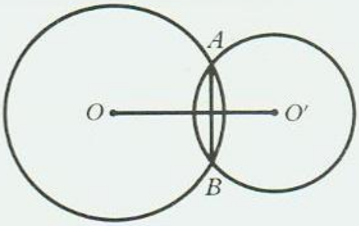


25. Given: $m\widehat{PM} = m\widehat{MQ}$
$\overline{XM} \perp \overline{OP}$
$\overline{YM} \perp \overline{OQ}$
Prove: $XM = YM$

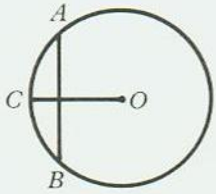


(*Hint:* Draw $\overline{OM}$ and use congruent triangles.)

26. Given: $\overline{AB}$ is a common chord to circles O and O'.
Prove: $\overline{OO'}$ is the perpendicular bisector of $\overline{AB}$.



27. Given: $\overline{OC}$ bisects $\widehat{ACB}$.
Prove: $\overline{OC}$ bisects $\overline{AB}$.

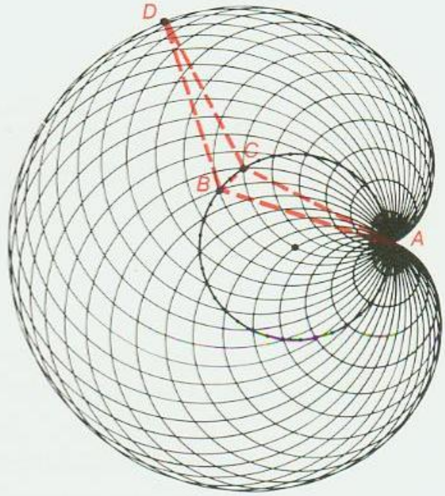


PROBLEM SOLVING

A cardioid has been drawn using the method described in the activity on page 348.

Points B and C are centers of circles through A that meet again at point D.

Why are $\triangle ABC$ and $\triangle DBC$ congruent?



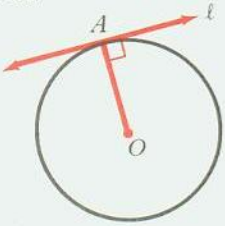
10–5 Tangents to Circles

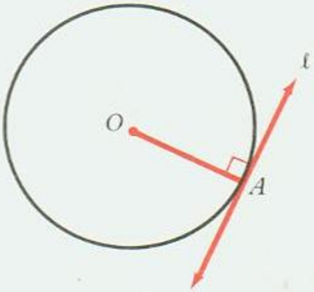
REVIEW: A line is tangent to a circle if it intersects the circle in exactly one point.

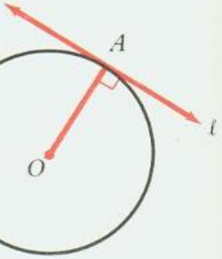
Suppose you want to round off the corners of a piece of wood in order to make a small table. In order to do a neat job a way must be found to draw a circular arc. The edges of the board must be tangent to the circular arc. How can this arc be drawn?

A theorem in this lesson helps solve the problem.

In each of these figures $\overline{OA}$ is a radius and ℓ is perpendicular to $\overline{OA}$ at A.







Is ℓ a tangent line?

Your observations suggest the following theorem.

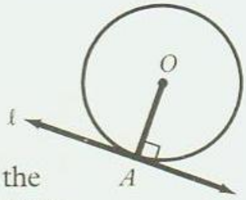
Theorem 10–8 If a line is perpendicular to a radius at a point on the circle, then the line is tangent to the circle.

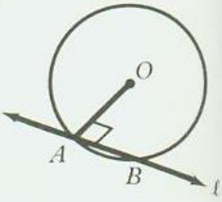
PROOF

Given: $\ell \perp \overline{OA}$

Prove: ℓ is tangent to the circle.

Plan: Use an indirect proof. Assume ℓ is not tangent to the circle. This means ℓ does not intersect the circle or ℓ intersects the circle in two places. We will investigate the latter assumption.



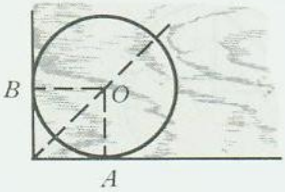


Statements	Reasons
1. ℓ intersects the circle at a second point B.	1. Indirect proof assumption
2. $\overline{OA} \perp \ell$	2. Given
3. $\overline{OB}$ is a hypotenuse of a right triangle.	3. Definition of hypotenuse
4. $OB > OA$	4. Length of the hypotenuse is greater than the length of either side.
5. $OB = OA$	5. Definition of circle

Statements 4 and 5 are contradictory. Hence the assumption is false and the line ℓ is tangent to the circle.

APPLICATION

We now solve the problem posed at the beginning of the lesson.



Step 1 Draw the angle bisector of the angle.

Step 2 Select a point O on the angle bisector and draw the perpendiculars $\overline{OA}$ and $\overline{OB}$ to the sides of the angle.

Step 3 Point O on the angle bisector is equidistant from the sides of the angles. Therefore $OA = OB$. Draw the circle centered at O through points A and B.

The edges of the board are tangent to the circle because of Theorem 10–8.

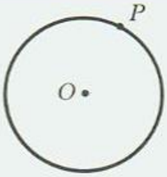
Here are two other theorems about tangents.

Theorem 10–9 If a line is tangent to a circle, then the radius drawn to the point of contact is perpendicular to the tangent.

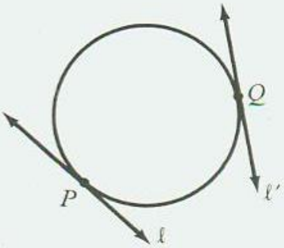
Theorem 10–10 If a line is perpendicular to a tangent at a point on the circle, then the line contains the center of the circle.

EXERCISES

A.



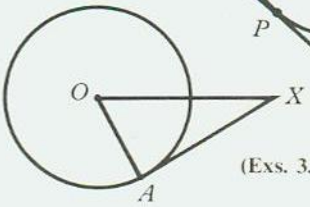
1. Draw a circle O and mark a point P on the circle. Construct a tangent to the circle through P.



2. Trace the figure at the right in which ℓ and ℓ' are tangents at points P and Q, respectively. Find the center of the circle. (*Hint:* Use Theorem 10–10.)

3. $\overleftrightarrow{AX}$ is tangent to the circle at A. $m\angle AOX = 51$. What is $m\angle AXO$?

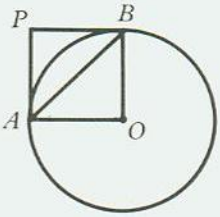
4. $\overleftrightarrow{AX}$ is tangent to the circle at A. If $OA = 10$ and $AX = 24$, find OX.



(Exs. 3, 4)

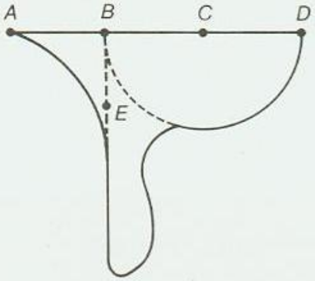
B.

5. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
$\overline{OA}$ and $\overline{OB}$ are radii each 4 cm in length.
$\overline{PA} \perp \overline{PB}$
Prove: $AOBP$ is a square.



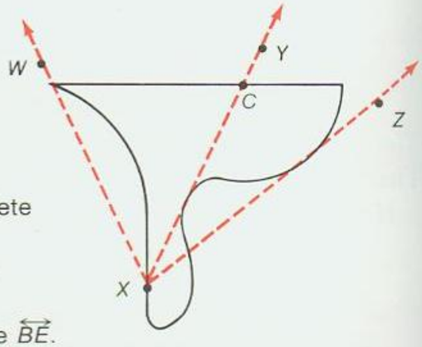
Activity

Draw, label, and cut out of cardboard this figure, sometimes called "the tomahawk."



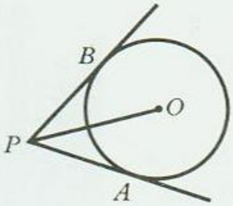
1. Draw $\overline{AD}$ so that $AB = BC = CD$.
2. Draw a semicircle on $\overline{BD}$.
3. Draw $\overline{BE} \perp \overline{AD}$ and complete the figure as shown.
4. Position the tomahawk on $\angle WXY$ so that
   a. X is on the handle edge $\overleftrightarrow{BE}$.
   b. A is on one ray of the angle.
   c. The semicircular edge of the tomahawk is tangent to the other edge of the angle.
5. Then $\overrightarrow{XC}$ is the angle trisector.

Use a tomahawk to trisect several angles.

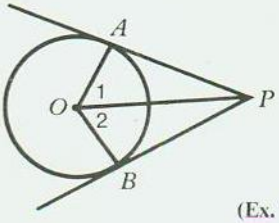


$m\angle YXZ = \frac{1}{3}m\angle WXZ$

6. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
$\overline{PA} \perp \overline{PB}$
$OB = 8$
Find: OP

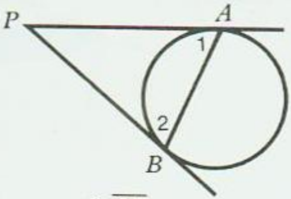


7. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
$\overline{OA}$ and $\overline{OB}$ are radii.
Prove: $\angle 1 \cong \angle 2$



(Ex. 7)

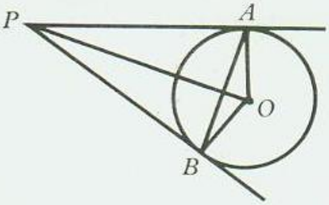
8. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
Prove: $m\angle 1 = m\angle 2$



C.

9. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
$\overline{OA}$ and $\overline{OB}$ are radii.
Prove: $\overline{OP}$ is the perpendicular bisector of $\overline{AB}$.

10. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
$\overline{OA}$ and $\overline{OB}$ are radii.
$m\angle APB = 80$
Find: $m\angle ABO$ (Hint: Use the conclusion of exercise 9.)

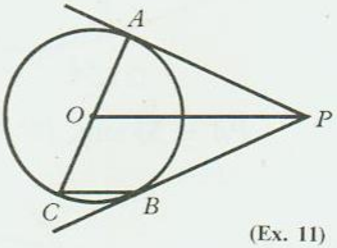


(Exs. 9, 10)

11. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents to circle O.
$\overline{AC}$ is a diameter.
Prove: $\overline{OP} \parallel \overline{BC}$.

12. Prove Theorem 10–9.

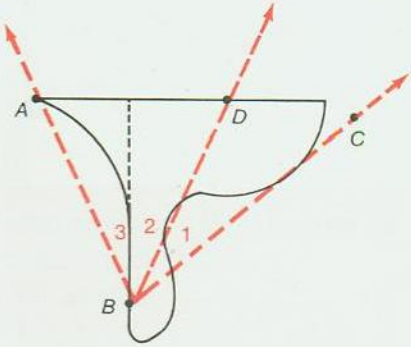
13. Prove Theorem 10–10.



(Ex. 11)

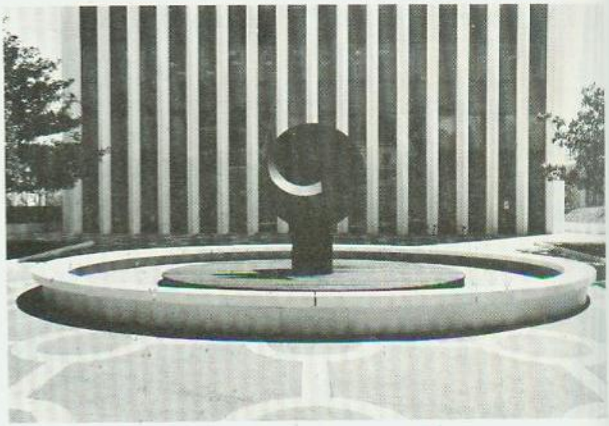
PROBLEM SOLVING

Prove that the tomahawk method in the last activity does work by proving that ∠1, ∠2, and ∠3 are congruent to each other.

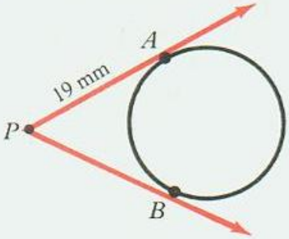


10–6 Tangents from a Point to a Circle

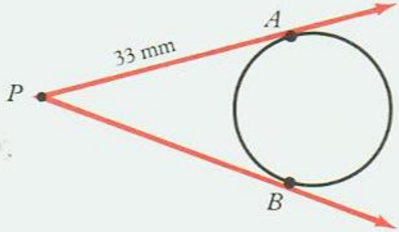
A surveyor is asked to find the center of a large circular wall of a fountain. A surveyor's pole and transits are available. A theorem in this lesson provides one method of finding the center using this equipment.



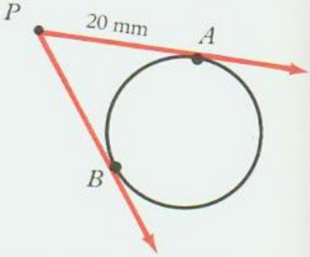
In each case, $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangents at A and B. Measure with ruler or protractor to fill in each unknown length.



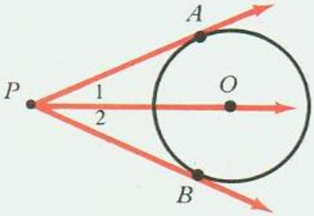
$PA = 19$ mm, $PB = \underline{?}$



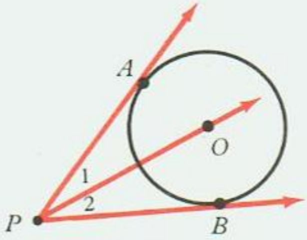
$PA = 33$ mm, $PB = \underline{?}$



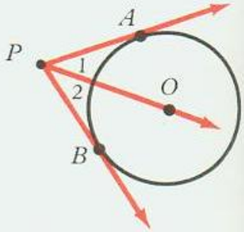
$PA = 20$ mm, $PB = \underline{?}$



$m\angle 1 = 24$, $m\angle 2 = \underline{?}$



$m\angle 1 = 25$, $m\angle 2 = \underline{?}$

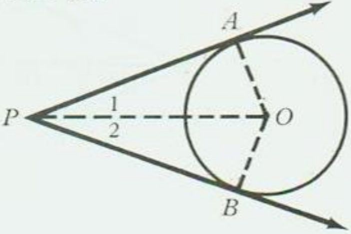


$m\angle 1 = 38$, $m\angle 2 = \underline{?}$

Theorem 10–11 The tangent segments to a circle from a point outside the circle are congruent and form congruent angles with the line joining the center and the point.

PROOF

Given: $\overrightarrow{PA}$ and $\overrightarrow{PB}$ tangent at A and B
Prove: $\overline{PA} \cong \overline{PB}$ and $\angle 1 \cong \angle 2$



Statements	Reasons
1. Draw $\overrightarrow{PO}$ and radii $\overline{OA}$ and $\overline{OB}$.	1. Construction
2. $OA = OB$	2. Definition of radius
3. $PO = PO$	3. Why?
4. $\overline{OA} \perp \overrightarrow{PA}$ and $\overline{OB} \perp \overrightarrow{PB}$	4. Why?
5. $\triangle POA \cong \triangle POB$	5. Why?
6. $\overline{PA} \cong \overline{PB}$, $\angle 1 \cong \angle 2$	6. CPCTC

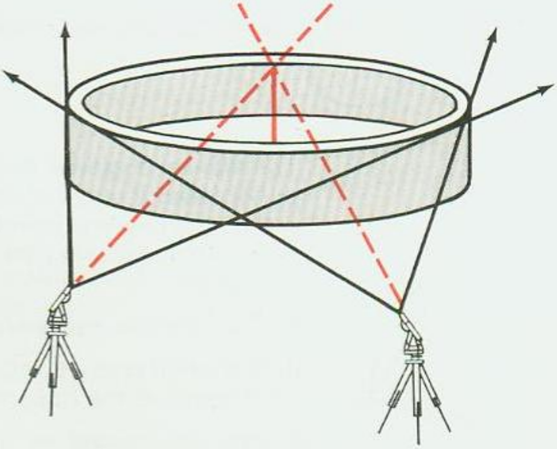
APPLICATION

The surveyor's problem introduced at the beginning of the lesson can be solved using Theorem 10-11 and the fact that the bisector of an angle is unique.

Step 1 Set up two transits and determine in each case the position of the tangent rays.

Step 2 Set the telescope of both transits in the position of the bisector of the angle formed by the tangents.

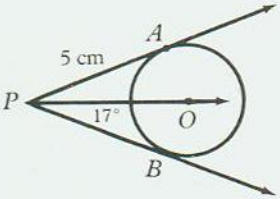
Step 3 A pole placed in the "line of vision" of both telescopes must be in the center of the circle.



NOTE: Not drawn to scale

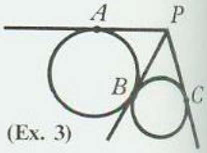
EXERCISES

A.



Given: $\overline{PA}$ and $\overline{PB}$ are tangents.
$PA = 5$ cm, $m\angle BPO = 17$

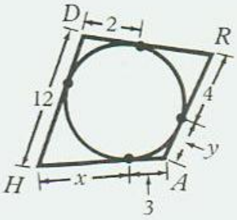
1. Find PB.
2. Find $m\angle APB$.

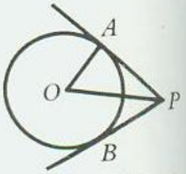


(Ex. 3)

3. **Given:** $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ are tangents.
$PA = 10$ cm
Find: PC.

B.

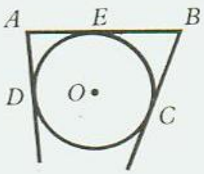




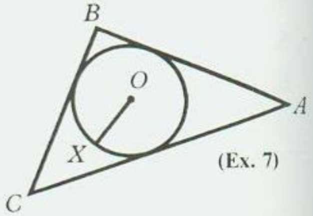
(Ex. 5)

4. **Given:** $\overline{HA}$, $\overline{AR}$, $\overline{RD}$, and $\overline{DH}$ are tangents.
Find: Lengths x and y.

5. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents to $\odot O$.
$OA = 10$
$m\angle APB = 60$
Find: OP.



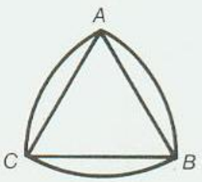
6. **Given:** $\overline{AB}$, $\overline{AD}$, and $\overline{BC}$ are tangents to $\odot O$.
Prove: $AD + BC = AB$

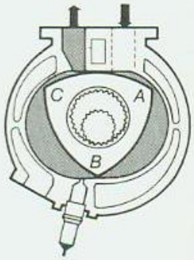


(Ex. 7)

7. **Given:** Right triangle ABC
$\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are tangents.
$AB = 6$, $BC = 8$ and $AC = 10$
Find: The length of radius $\overline{OX}$

Activity

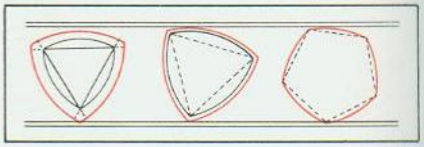




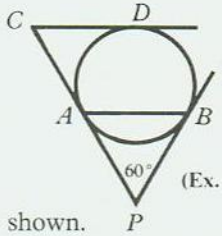
The Wankel engine is designed around a curve called a curve of constant width.

Construct several identical curved shapes from cardboard using the following procedure.

1. Construct an equilateral triangle $\triangle ABC$.
2. Construct arcs $\widehat{AB}$, $\widehat{AC}$, and $\widehat{BC}$ each centered at the opposite vertex.
3. Use these curves as "rollers" as shown in this diagram to demonstrate that they have the same width.



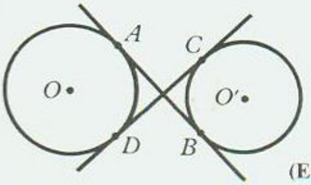
8. Given: $\overline{CP}$, $\overline{CD}$, and $\overline{PB}$ are tangents.
$CD = 4$, $CP = 9$
$m\angle APB = 60$
Find: AB.



(Ex. 8)

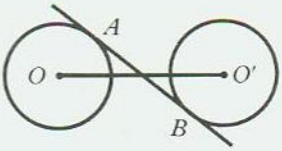
C.

9. Given: $\overline{AB}$ and $\overline{CD}$ are common tangents as shown.
Prove: $AB = CD$.

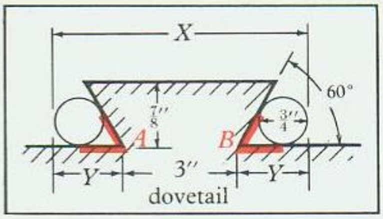


(Ex. 9)

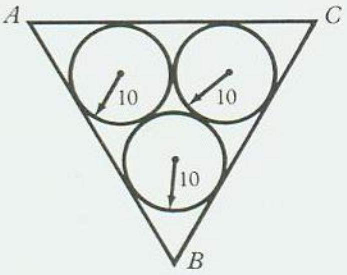
10. Given: Circle $O \cong$ circle O'
$\overline{AB}$ is a common internal tangent.
Prove: $\overline{AB}$ bisects $\overline{OO'}$.



11. Quality control in the manufacturing of machine parts often requires unusual methods of measurement. For example, in order to check that the angles A and B are correct on a part called a dovetail, circular plugs are inserted as shown. Then the distance X is measured with a micrometer. For the dovetail shown here, what should this distance equal?



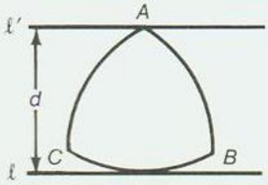
12. Three metal discs each with a radius of 10 cm are tangent to each other. The discs are enclosed by a metal frame that forms an equilateral triangle. What is the length of one side of the frame?



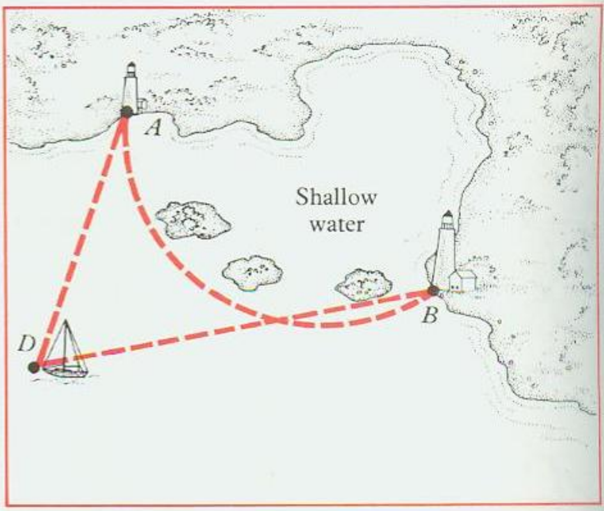
PROBLEM SOLVING

Suppose ℓ is a line tangent to an arc of the curve constructed in the Activity on page 366 and ℓ' is parallel to ℓ through an opposite vertex.

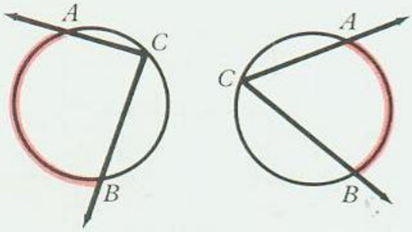
How does the distance d between these lines compare to length AB?



10–7 Measures of Inscribed Angles



Two lighthouses can serve as a navigation aid for a ship sailing near coastlines and shallow water. If a ship is positioned at point D, then $\angle ADB$ should be less than a published "danger angle." This navigation technique is based upon a theorem developed in this section.

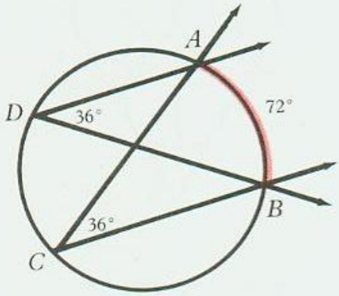


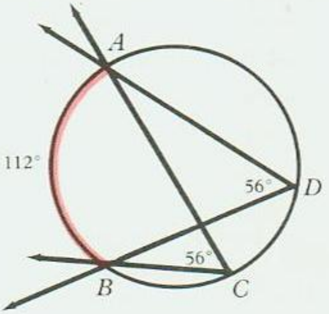
An inscribed angle determines an arc called the *intercepted arc.*

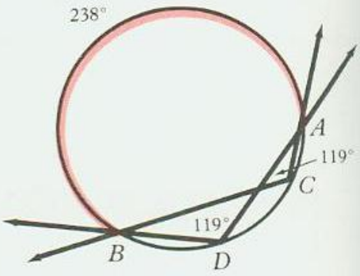
Definition 10–12

The **intercepted arc** for an inscribed $\angle ACB$ is the arc AB lying in the interior of the angle.

In each figure $m\widehat{AB}$ and $m\angle ACB$ and $m\angle ADB$ are given.







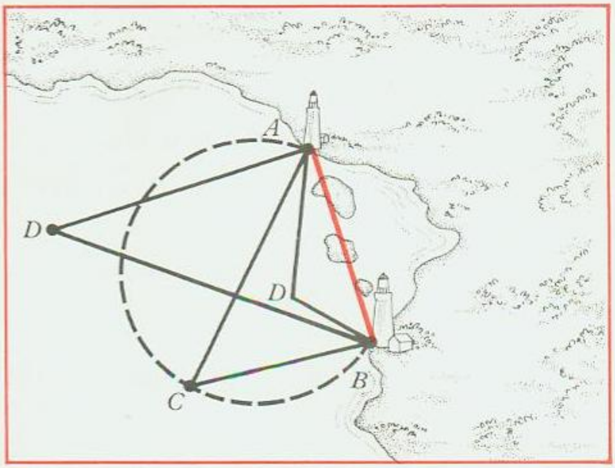
These figures suggest the following theorem.

Theorem 10–12 The measure of an inscribed angle is one half the measure of its intercepted arc.

APPLICATION 1

The navigation technique on page 368 is based upon Theorem 10–12. If a point C lies on a circle such that arc AB has measure twice the "danger angle," then $m\angle ACB$ equals the danger angle.

If D were on this same circle or inside it, then $m\angle ADB$ would be equal to or greater than the danger angle. When D is outside the circle, $m\angle ADB$ is less than the danger angle and a ship at point D is in a safe position.



The second application is based upon a special case of Theorem 10–12, which is stated as Theorem 10–13.

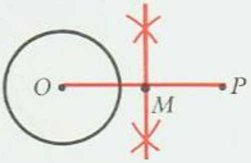
APPLICATION 2

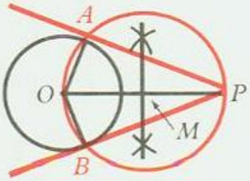
A draftsman often needs to draw from a given point outside a circle two tangent lines to the circle. Here is one way that it can be done.

Step 1 From a point P outside a given circle with center O, draw $\overline{OP}$ and its midpoint M.

Step 2 Draw the circle with diameter $\overline{OP}$ intersecting the given circle at points A and B. Draw $\overrightarrow{PA}$ and $\overrightarrow{PB}$. Theorem 10–12 tells us that $\angle OAP$ and $\angle OBP$ are right angles.

Step 3 Theorem 10-8 tells us that $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangent to the given circle.





Theorem 10–13 An angle inscribed in a semicircle is a right angle.

EXERCISES

A.

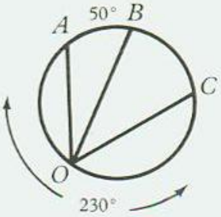
In exercises 1-4 $m\widehat{AB} = 50$, $m\widehat{AOC} = 230$.

1. $m\angle AOB = \underline{\ ?\ }$

2. $m\angle BOC = \underline{\ ?\ }$

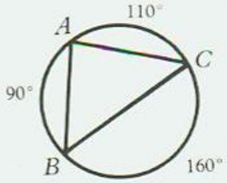
3. $m\angle AOC = \underline{\ ?\ }$

4. $m\widehat{AC} = \underline{\ ?\ }$

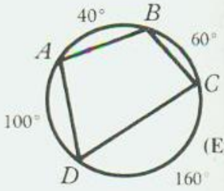


(Exs. 1-4)

5. Find the measures of each angle of $\triangle ABC$.



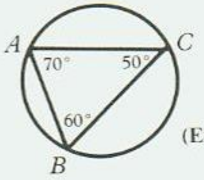
(Ex. 5)



(Ex. 6)

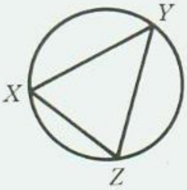
6. Find the measure of each angle of $ABCD$.

7. Find $m\widehat{AB}$, $m\widehat{AC}$, and $m\widehat{BC}$.



(Ex. 7)

8. Given: $XY = YZ$
$m\angle Y = 40$
Find: $m\widehat{XY}$, $m\widehat{YZ}$, and $m\widehat{XZ}$



For exercises 9–14 $m\widehat{AC} = 160$, $m\widehat{AB} = 75$, $m\widehat{CD} = 45$.

9. $m\angle ABC = \underline{\ ?\ }$

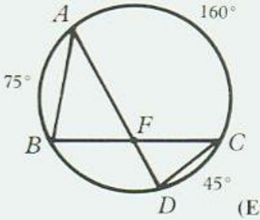
10. $m\angle ADC = \underline{\ ?\ }$

11. $m\widehat{BD} = \underline{\ ?\ }$

12. $m\angle BAD = \underline{\ ?\ }$

13. $m\angle BCD = \underline{\ ?\ }$

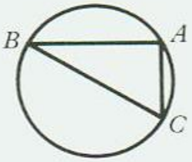
14. $m\angle AFB = \underline{\ ?\ }$



(Exs. 9–14)

B.

15. Given: $\overline{BC}$ is a diameter.
$AB = 8$, $AC = 6$
Find: BC



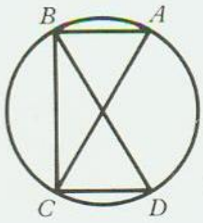
For exercises 16–19, $\overline{AC}$ is a diameter, $m\widehat{CD} = 66$, and $m\angle CDB = 60$.

16. $m\angle ADB = \underline{\ ?\ }$

17. $m\widehat{BC} = \underline{\ ?\ }$

18. $m\angle BCA = \underline{\ ?\ }$

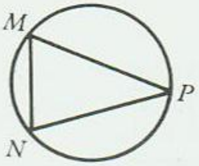
19. $m\angle BCD = \underline{\ ?\ }$



(Exs. 16–19)

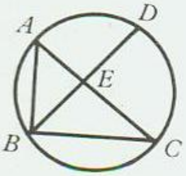
20. Given: $\frac{m\widehat{MN}}{m\widehat{NP}} = \frac{2}{3}$ and $\frac{m\widehat{NP}}{m\widehat{MP}} = \frac{3}{4}$

Find: $m\angle M$, $m\angle N$, and $m\angle P$.



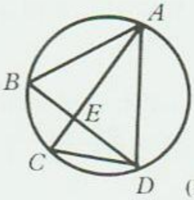
21. Given: $m\widehat{BC} = 100$, $m\widehat{AB} = 80$
$\overline{AC}$ is a diameter.
$\overline{BD} \perp \overline{AC}$

Find: **a.** $m\angle BAC$ **b.** $m\angle ACB$
c. $m\angle EBC$ **d.** $m\widehat{AD}$



22. Given: $\overline{AC}$ bisects $\angle BAD$.
$m\widehat{CD} = 80$, $m\widehat{AD} = 160$

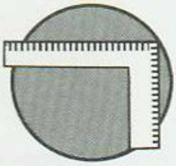
Find: **a.** $m\angle BAC$ **b.** $m\angle BDC$
c. $m\angle AEB$ **d.** $m\angle ADB$



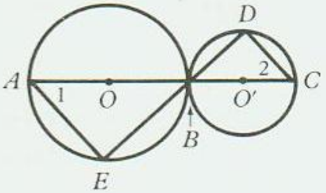
(Exs. 22, 23)

23. Given: $\overline{AC}$ bisects $\angle BAD$.
Prove: Three angles of $\triangle ABE$ are congruent to three angles of $\triangle DCE$.

24. How can a carpenter's square be used to find the center of a circular disc? Why does your method work?

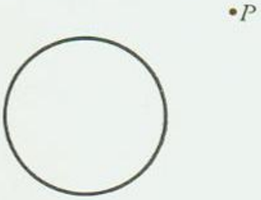


25. Given: $\overline{AB}$ is a diameter of circle O.
$\overline{BC}$ is a diameter of circle O'.
Circle O is tangent to circle O' at B.
Prove: $m\angle 1 = m\angle 2$.



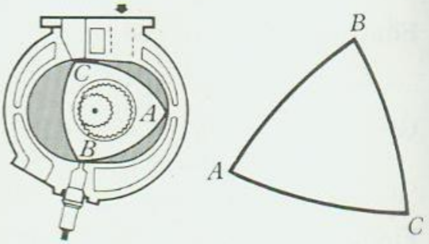
26. Prove that all inscribed angles with the same or congruent intercepted arcs are equal in measure.

27. Draw a circle and a point P not in or on the circle. Construct two lines through P that are tangent to the circle.



C.

28. A curve of constant width like the one in a Wankel engine consists of three circular arcs, AB, BC, and AC. What is the measure of each of these arcs? (Review the construction shown in the activity on p. 366.)

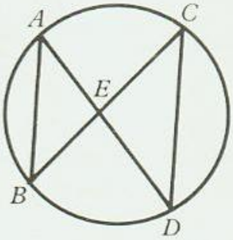


29. Prove that if a quadrilateral is inscribed in a circle, its opposite angles are supplementary.

30. If $ABCD$ is a quadrilateral inscribed in a circle and $m\angle A = 3x + 50$, $m\angle B = 4x + 25$, and $m\angle C = 7x + 30$, find $m\angle D$.

31. Prove that a parallelogram inscribed in a circle is a rectangle.

32. In the figure $\angle BAD$ and $\angle BCD$ are inscribed angles. Prove that $\triangle ABE$ is similar to $\triangle CDE$.



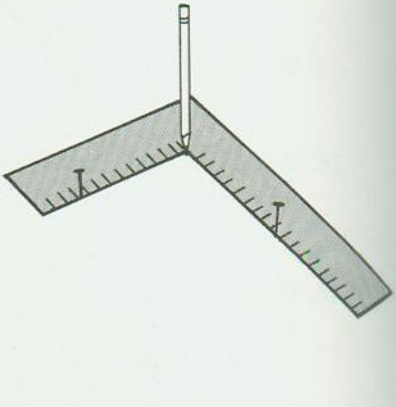
(Ex. 32)

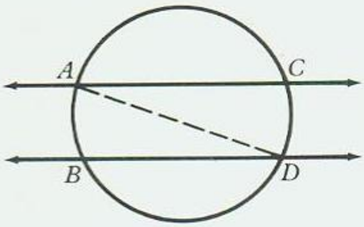
Activity

A carpenter's square can be used to draw a circle.

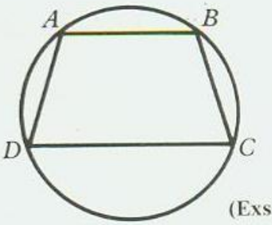
1. Hold the square against a pair of nails and a pencil at the right angle.
2. Rotate the square, always keeping the arms of the square against the nails. The pencil will draw the semicircle with diameter determined by the nails. Why?

Draw a circle using this method.





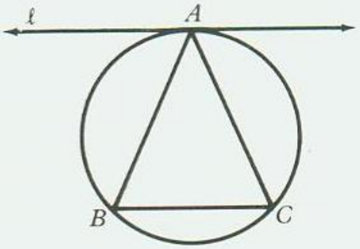
33. Prove that if two parallel lines intersect a circle, then they intercept congruent arcs.



(Exs. 34, 35)

34. **Given:** $ABCD$ is an inscribed trapezoid.
Prove: $ABCD$ is an isosceles trapezoid.

35. **Given:** $ABCD$ is an inscribed quadrilateral and $\overline{AD} \cong \overline{BC}$.
Prove: $ABCD$ is a trapezoid.

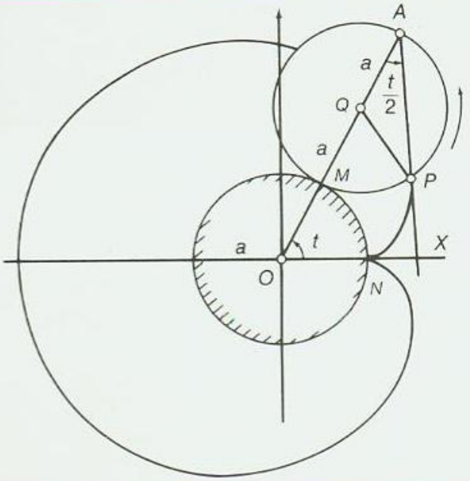


36. **Given:** ℓ is tangent to the circle at A.
$\overline{BC} \parallel \ell$
Prove: $\triangle ABC$ is an isosceles triangle.

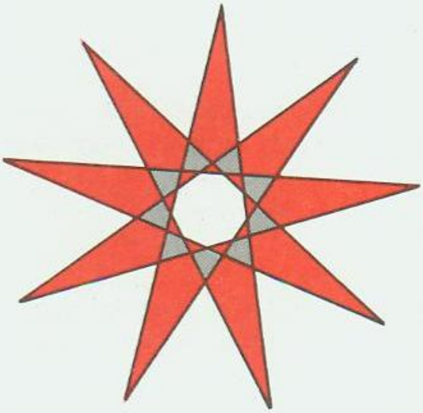
PROBLEM SOLVING

The figure at the right shows one circle rolling around another circle and generating a cardioid.

If $m\angle AOX = t$, why is $m\angle OAP = \frac{t}{2}$?

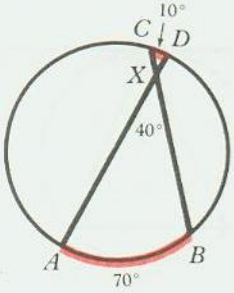


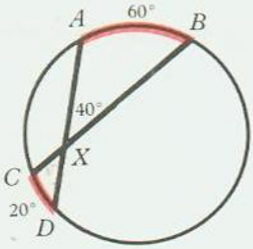
10–8 Angles Formed by Chords

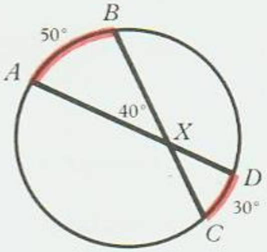


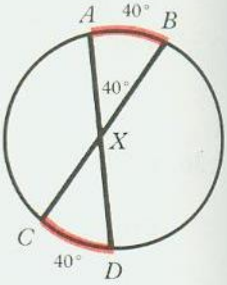
The star polygon is drawn by joining every fourth point from nine evenly spaced points on a circle. (See Problem Solving, p. 345.) There are many angles in this star design that appear to be congruent. In this lesson we study a theorem that can be used to prove these angles congruent.

In each figure $m\widehat{AB} + m\widehat{CD} = 80$.









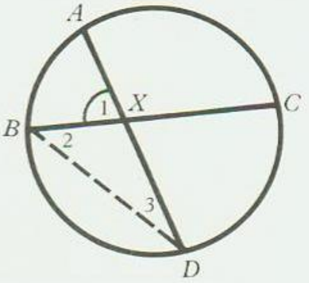
These figures suggest the following theorem.

Theorem 10–14 An angle formed by two chords intersecting inside a circle has a measure equal to one half the sum of the intercepted arcs.

PROOF

Given: Chords $\overline{AD}$ and $\overline{BC}$ intersecting in point X

Prove: $m\angle AXB = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$

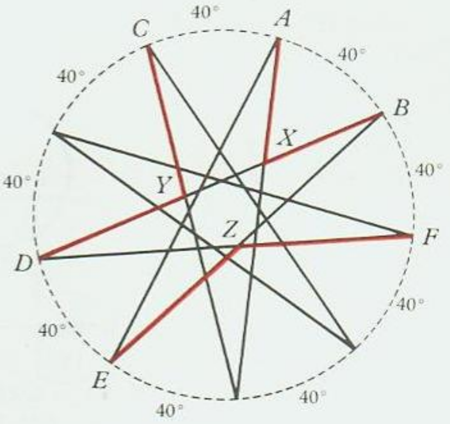


Statements	Reasons
1. Construct $\overline{BD}$.	1. Construction
2. $m\angle 2 = \frac{1}{2}m\widehat{CD}$	2. The measure of an inscribed angle is one half its intercepted arc.
3. $m\angle 3 = \frac{1}{2}m\widehat{AB}$	3. Why?
4. $m\angle AXB = m\angle 2 + m\angle 3$	4. Why?
5. $m\angle AXB = \frac{1}{2}m\widehat{AB} + \frac{1}{2}m\widehat{CD}$	5. Substitution (Statements 2, 3, 4)
6. $m\angle AXB = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$	6. Distributive Property

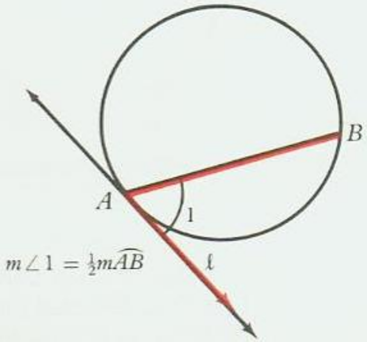
APPLICATION

Determine the angle measures of angles in this star polygon. We shall use Theorem 10–14.

1. $m\angle AXB = \frac{1}{2}(40 + 80) = 60$
2. $m\angle CYD = \frac{1}{2}(80 + 120) = 100$
3. $m\angle EZF = \frac{1}{2}(120 + 160) = 140$



There is a special case of Theorem 10–14 that involves a tangent line. It is stated below.

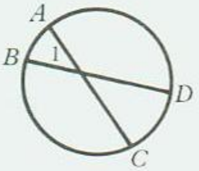


Theorem 10–15 The measure of an angle formed by a tangent and a chord drawn to the point of contact is one half the intercepted arc.

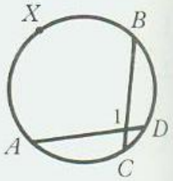
EXERCISES

A.

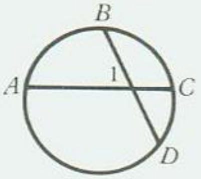
1. **Given:** $m\overset{\frown}{AB} = 30$
   $m\overset{\frown}{CD} = 20$
   Find: $m\angle 1$



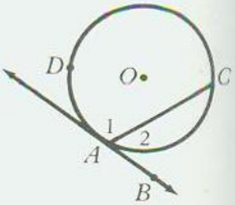
2. **Given:** $m\overset{\frown}{AXB} = 190$
   $m\overset{\frown}{CD} = 25$
   Find: $m\angle 1$



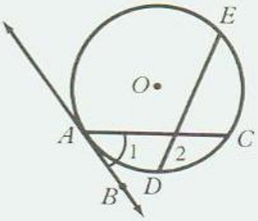
3. **Given:** $m\angle 1 = 80$
   $m\overset{\frown}{AB} = 100$
   Find: $m\overset{\frown}{CD}$



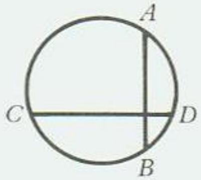
4. **Given:** $\overleftrightarrow{AB}$ is tangent to circle O.
   $m\overset{\frown}{ADC} = 300$
   Find: $m\angle 1$ and $m\angle 2$



5. **Given:** $\overleftrightarrow{AB}$ is tangent to circle O.
   $m\overset{\frown}{AE} = 160$, $m\overset{\frown}{AD} = 50$, $m\overset{\frown}{DC} = 60$
   Find: $m\angle 1$ and $m\angle 2$

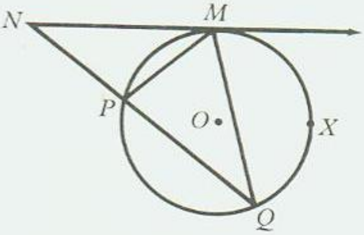


6. **Given:** $\overline{AB} \perp \overline{CD}$
   $m\overset{\frown}{BD} = 20$
   $m\overset{\frown}{AD} = 80$
   Find: $m\overset{\frown}{AC}$ and $m\overset{\frown}{BC}$.



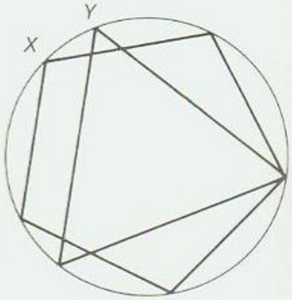
B.

7. **Given:** $\overleftrightarrow{MN}$ is tangent to circle O.
   $m\overset{\frown}{PQ} = 100$, $m\overset{\frown}{MXQ} = 150$
   Find: **a.** $m\angle NMP$ **b.** $m\angle PQM$
   c. $m\angle MPQ$ **d.** $m\angle PMQ$
   e. $m\angle MNP$

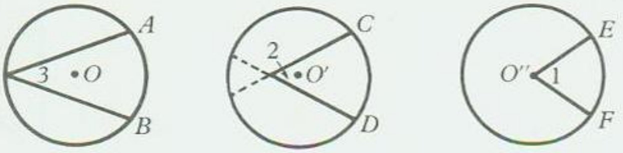


Activity

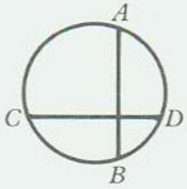
Using a compass and straightedge construct 15 points evenly spaced around a circle. Then draw two star polygons, one by joining every fourth point and the other by joining every seventh point. (*Hint*: Construct a regular pentagon and an equilateral triangle with a common vertex. Then $m\overset{\frown}{XY} = 24 = \frac{1}{15}(360)$. The method for constructing a regular pentagon is described on page 274.)



8. The three circles shown are all congruent, with $m\widehat{AB} = m\widehat{CD} = m\widehat{EF}$. $\angle 1$ is a central angle and $\angle 3$ is an inscribed angle. Which of the three angles is the largest? the smallest? Why?

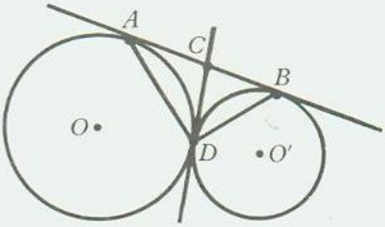


9. **Given:** $\overline{AB} \perp \overline{CD}$
Prove: $m\widehat{AD} + m\widehat{BC} = m\widehat{AC} + m\widehat{BD}$

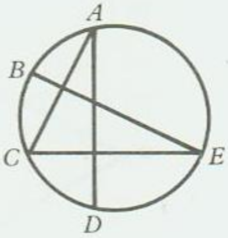


C.

10. **Given:** $\overline{AB}$ is a common external tangent to circles O and O'. $\overline{CD}$ is a common internal tangent.
Prove: $\angle ADB$ is a right angle.



11. **Given:** $\overline{AD} \perp \overline{CE}$
$\overline{BE} \perp \overline{AC}$
Prove: $m\widehat{BC} = m\widehat{CD}$.



PROBLEM SOLVING

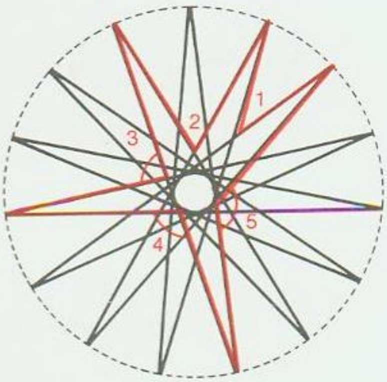
In the star polygon at the right five angles have been highlighted in color.

Without using a protractor find each of the following:

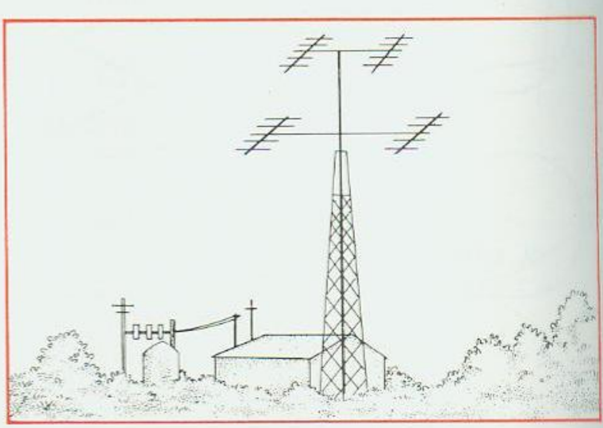
$m\angle 1 = \underline{?}$ $m\angle 4 = \underline{?}$

$m\angle 2 = \underline{?}$ $m\angle 5 = \underline{?}$

$m\angle 3 = \underline{?}$



10–9 Angles and Segments Formed by Tangents and Secants

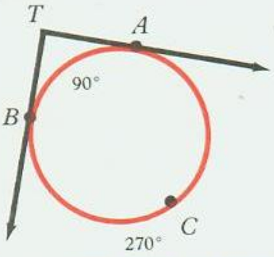


When engineers design radio towers, they need to know what fraction of the earth's surface will be covered by the radio beams from the tower.

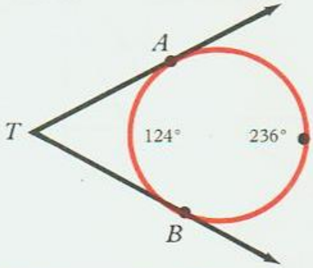
In this section we simplify the problem by considering a circular cross section of the earth through the base of the tower. We ask: If we know the measure of the angle formed by the top of the tower and the tangent rays to the circle, can we find the fraction of the circumference of the circle covered by the radio beams?

In each case $\overrightarrow{TA}$ and $\overrightarrow{TB}$ are tangent rays.

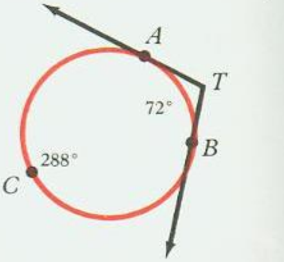
$m\widehat{ACB} - m\widehat{AB} = 180$



$m\widehat{ACB} - m\widehat{AB} = 112$

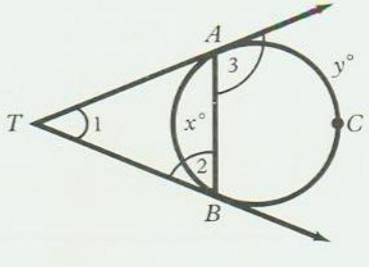


$m\widehat{ACB} - m\widehat{AB} = 216$



Measure $\angle ATB$ with a protractor. What do you find?

Theorem 10–16 The measure of an angle formed by two intersecting tangents to a circle is one half the difference of the measures of the intercepted arcs.



PROOF

Given: $\overrightarrow{TA}$ and $\overrightarrow{TB}$ are tangent rays to a circle, $m\widehat{AB} = x$, and $m\widehat{ACB} = y$.

Prove: $m\angle ATB = \frac{1}{2}(y - x)$

Statements	Reasons
1. $m\angle 2 = \frac{1}{2}x$	1. The measure of an angle formed by a tangent and a chord is one half the intercepted arc.
2. $m\angle 3 = \frac{1}{2}y$	2. Why?
3. $m\angle 3 = m\angle 1 + m\angle 2$	3. The measure of an exterior angle equals the sum of the measures of the two remote interior angles.
4. $m\angle 1 = m\angle 3 - m\angle 2$	4. Why?
5. $m\angle 1 = \frac{1}{2}y - \frac{1}{2}x$	5. Substitution
6. $m\angle 1 = \frac{1}{2}(y - x)$	6. Why?

APPLICATION

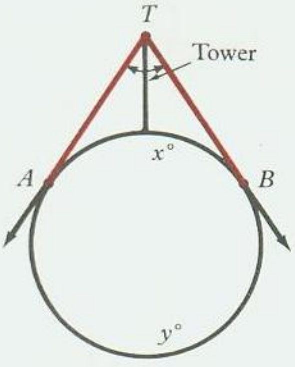
Suppose the angle formed by the two tangent rays from the top of the radio tower has a measure of 160°. What fraction of the circle do the radio waves cover?

Answer: 1. Theorem 10-16 results in equation (1) on the right and equation (2) expresses a property of a circle.

2. Solving the system of equations, we find that $x = 20$.

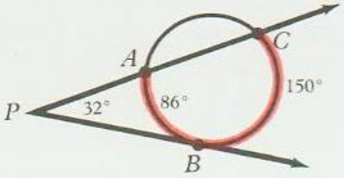
3. $$\frac{x}{360} = \frac{20}{360} = \frac{1}{18}$$

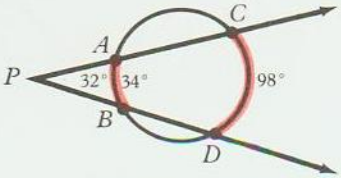
The radio waves cover $\frac{1}{18}$ of the circumference of the circle.



$$\frac{1}{2}(y - x) = 160 \quad (1)$$
$$(y + x) = 360 \quad (2)$$

These figures suggest an additional theorem.

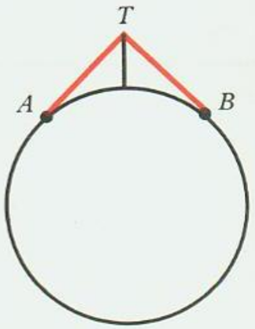


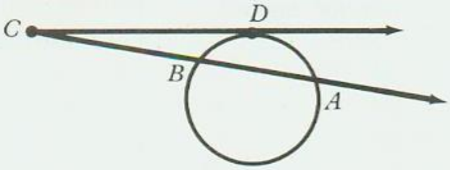


Theorem 10-17 The measure of an angle formed by a tangent and a secant or two secants from a point exterior to a circle is one half the difference of the measures of the intercepted arcs.

On page 378 we asked what fraction of the earth's surface will be covered by the radio beams from the tower. An equally important question is how far do the radio beams reach? The theorem on this page will give a good approximation to the answer.

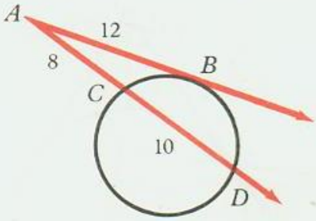
In order to proceed with the next theorem we need to introduce some more terms.



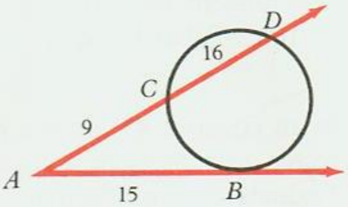


Recall that $\overleftrightarrow{AC}$ is a secant. We call $\overline{CA}$ a *secant segment.* $\overline{BC}$ is called an *external secant segment.* $\overline{CD}$ is a *tangent segment.*

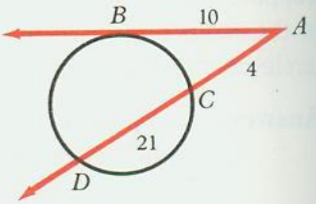
Consider the following examples of circles with a tangent and a secant. What relationship can you find that is common to all three examples?



Observe that
$12^2 = 8 \times 18$



Observe that
$15^2 = 9 \times 25$



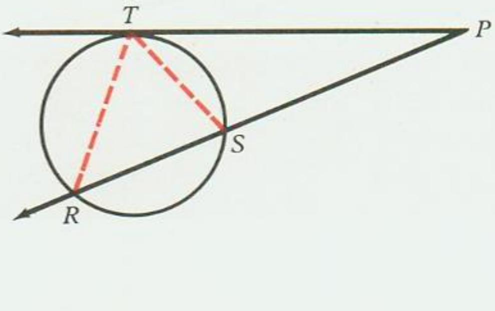
Observe that
$10^2 = 4 \times 25$

This relationship is stated as Theorem 10–18.

Theorem 10–18 If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the length of the tangent segment equals the product of the lengths of the secant segment and its external secant segment.

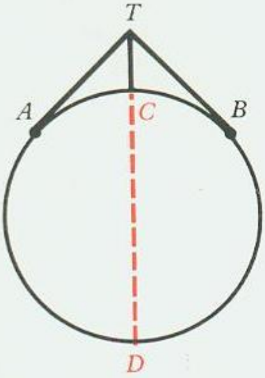
PROOF

Given: $\odot O$ with tangent segment $\overline{PT}$.
Prove: $(PT)^2 = PS \cdot PR$
Plan: Draw in $\overline{ST}$ and $\overline{TR}$. Use similar triangles.



Statements	Reasons
1. Draw $\overline{ST}$ and $\overline{TR}$	1. Construction
2. $\angle P \cong \angle P$	2. Reflexive property
3. $m\angle PTS = \frac{1}{2}m\widehat{TS}$	3. Why?
4. $m\angle SRT = \frac{1}{2}m\widehat{TS}$	4. Why?
5. $\angle PTS \cong \angle SRT$	5. Substitution, Definition of congruence
6. $\triangle PTS \sim \triangle PRT$	6. AA Similarity Theorem
7. $\frac{PT}{PR} = \frac{PS}{PT}$	7. Definition of similar triangles
8. $(PT)^2 = PS \cdot PR$	8. Theorem 9-1

APPLICATION



At the top of the previous page we asked how far the rays reach from the tower. The length TA is a good approximation. Theorem 10-18 says that $(TA)^2 = (TC)(TD)$ or $TA = \sqrt{(TC)(TD)}$.

Suppose the tower is 800 feet tall. We will assume that the diameter $\overline{CD}$ of the earth is 8000 miles × 5280 feet/mile or 42,240,000 feet. Then

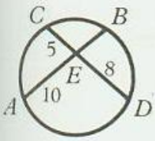
$$TA = \sqrt{800 \text{ ft} \times 42{,}240{,}800 \text{ ft}} \doteq 183{,}827.7 \text{ ft} \doteq 34.8 \text{ miles}.$$

The following two theorems also involve segments related to circles.

Theorem 10-19 If two chords intersect in a circle, then the product of the lengths of the segments of one chord equals the product of the lengths of the second chord.

Theorem 10-20 If two secant segments are drawn to a circle from an exterior point, then the product of the lengths of one secant segment and its external secant segment equals the product of the lengths of the other secant segment and its external secant segment.

Example 1 **Find:** BE

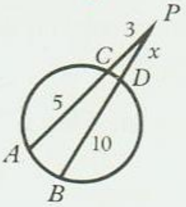


By Theorem 10-19,

$$CE \cdot ED = BE \cdot AE$$

$$BE = \frac{CE \cdot ED}{AE} = \frac{5 \cdot 8}{10} = 4$$

Example 2 **Find:** PD



By Theorem 10-20,

$$PC \cdot PA = PD \cdot PB$$

$$3 \cdot 8 = x(x + 10)$$

$$(x + 12)(x - 2) = 0$$

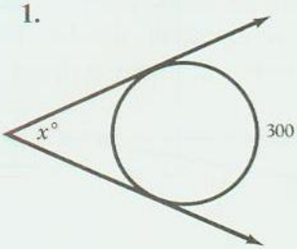
$$x = 2$$

EXERCISES

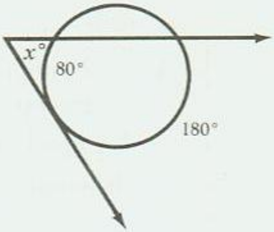
A.

In exercises 1–12, find x. You may assume that lines that appear to be tangents are tangents.

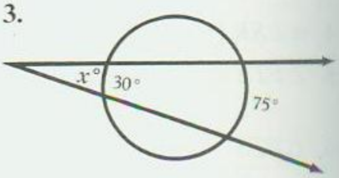
1.



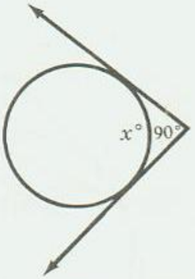
2.



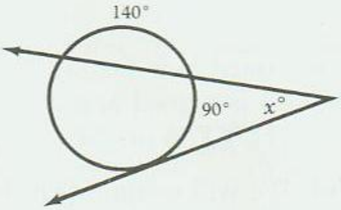
3.



4.



5.



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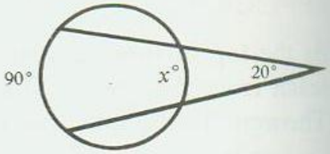

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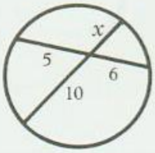

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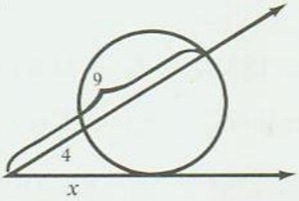

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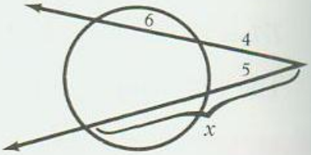
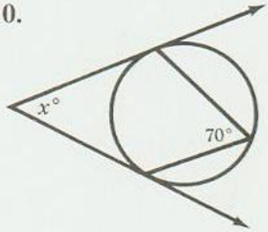
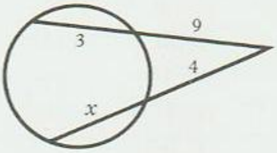


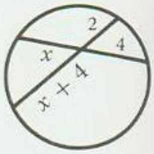
10.



11.

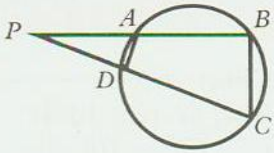


12.



B.

For exercises 13–15, $ABCD$ is an inscribed quadrilateral. $m\widehat{AB} = 100$, $m\widehat{AD} = 30$, $m\widehat{BC} = 90$.



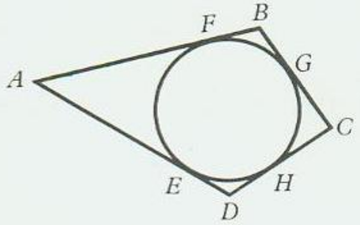
(Exs. 13–15)

13. Find $m\angle P$.

14. Find $m\angle BAD$.

15. Find $m\angle ABC$.

For exercises 16–19, *ABCD* is a circumscribed quadrilateral. $m\widehat{FG} = 60$, $m\widehat{GH} = 70$, $m\widehat{HE} = 80$.



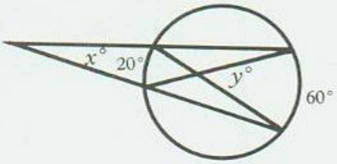
16. Find $m\widehat{EF}$.

17. Find $m\angle A$.

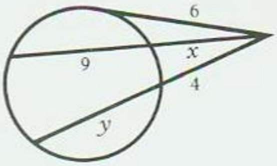
18. Find $m\angle B$.

19. Find $m\angle C$ and $m\angle D$.

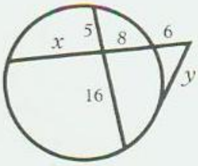
20. Find the values of x and y.



21. Find the values of x and y.



22. Find the values of x and y.

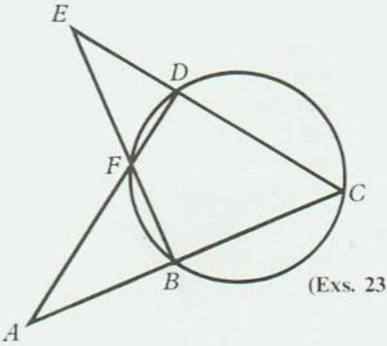


For exercises 23–25, $\overline{EC}$, $\overline{EB}$, $\overline{AD}$, and $\overline{AC}$ are all secants. $m\angle E = 40$, $m\widehat{BC} = 120$, and $m\widehat{BF} = 80$.

23. Find $m\widehat{DF}$.

24. Find $m\widehat{DC}$.

25. Find $m\angle A$.



(Exs. 23-25)

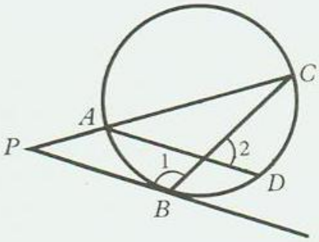
For exercises 26–29, $m\widehat{AB} = 55$, $m\widehat{BD} = 40$, $\overline{AC}$ is a diameter, and $\overline{PB}$ is tangent to the circle at *B*.

26. Find $m\angle P$.

27. Find $m\angle 2$.

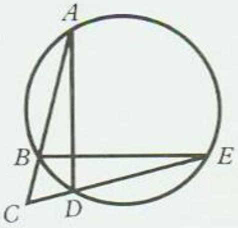
28. Find $m\angle 1$.

29. Is $\overline{AD} \parallel \overline{PB}$? Explain.

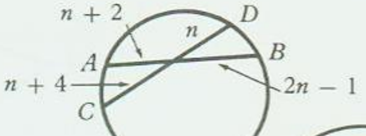


(Exs. 26-29)

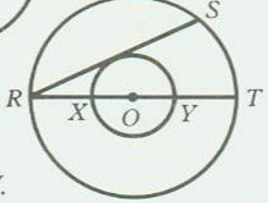
30. Given: $\overline{AD} \perp \overline{BE}$ and $m\angle C = 40$
Find: $m\widehat{BD}$



31. Find AB and CD.

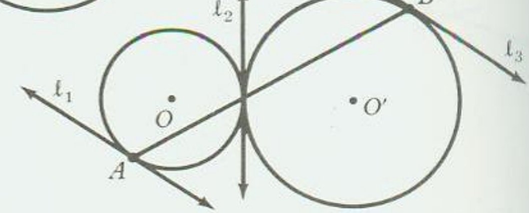


C.



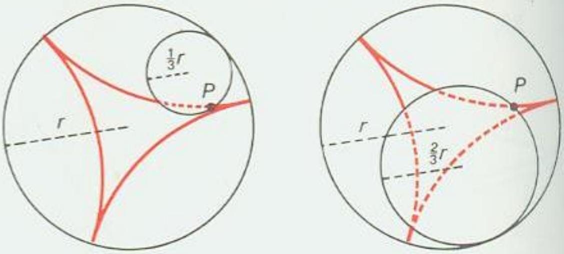
32. In the figure, O is the center of both circles. $\overline{RS}$ is a tangent to the smaller circle. If $RX = 5$ and $RS = 30$, find XY.

33. In this figure ℓ_2 is tangent to both circles, ℓ_1 and ℓ_3 are tangent to the circles at A and B respectively. Prove that $\ell_1 \parallel \ell_3$.

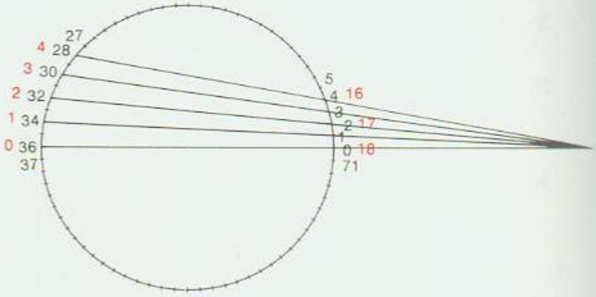


Activity

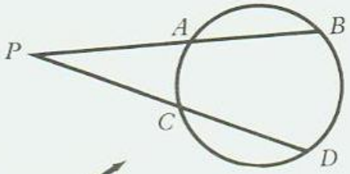
When a circle is rolled along the inside of a larger circle, a point P on the rolling circle moves on a path (shown in red) called a *deltoid,* provided that the radius of the rolling circle is $\frac{1}{3}$ or $\frac{2}{3}$ the larger radius. Draw a deltoid as outlined below.



1. Draw a circle with a 3 cm radius in the middle of a notebook size sheet of paper. Mark points at 5° intervals. Number the points 0, 1, 2, . . . in a counterclockwise manner (shown in black.)
2. Begin with 36 (shown in black) and number alternate points 0, 1, 2, . . . in a clockwise manner (shown in red.)
3. Draw a ray *from* a point with a *red* number *through* a point with the same number shown in *black*. Do this for all numbers 0 through 71.
4. Outline the deltoid figure surrounding the circle.



34. If $\overline{PB}$ and $\overline{PD}$ are secant segments and $PB = PD$, prove that $PA = PC$.

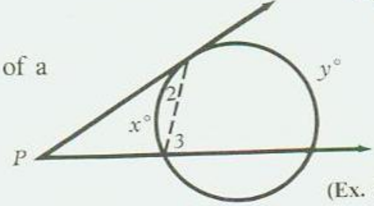


(Ex. 34)

35. Prove Theorem 10–17 for the case of a tangent and a secant.

36. Prove Theorem 10–19.

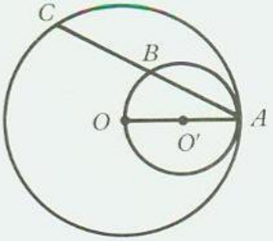
37. Prove Theorem 10–20.



(Ex. 35)

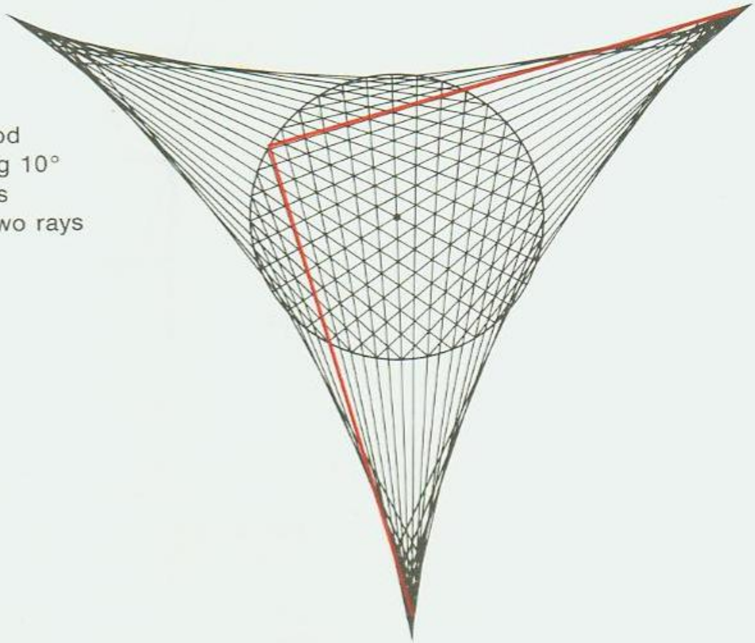
38. If two circles are tangent internally and the diameter of the smaller circle is the radius of the larger circle, then any chord of the larger circle drawn to the point of tangency will be bisected by the smaller circle.

Given: $O'A = \frac{1}{2}OA$ and the circles are tangent at A.
Prove: B is the midpoint of $\overline{AC}$.



PROBLEM SOLVING

A deltoid is drawn with the method described in the last activity using 10° intervals. Each point has two rays emanating from it. Why are the two rays perpendicular to each other?



Important Ideas—Chapter 10

Terms

Radius (p. 342)
Chord (p. 342)
Diameter (p. 342)
Tangent (p. 343)
Secant (p. 343)
Inscribed angle (p. 343)
Central angle (p. 343)
Minor arc (p. 346)
Major arc (p. 346)
Measure of a minor arc (p. 346)
Measure of a major arc (p. 346)
Congruent arcs (p. 347)
Congruent circles (p. 347)

Postulate

Arc Addition Postulate (p. 346)

Theorems

10–1 In a circle or in congruent circles congruent chords have congruent minor arcs.

10–2 In a circle or in congruent circles congruent minor arcs have congruent chords.

10–3 In a circle or in congruent circles congruent chords are equidistant from the center.

10–4 In a circle or in congruent circles chords equidistant from the center are congruent.

10–5 The perpendicular bisector of a chord contains the center of the circle.

10–6 If a line through the center of a circle is perpendicular to a chord that is not a diameter, then it bisects the chord and its minor arc.

10–7 If a line through the center of a circle bisects a chord that is not a diameter, then it is perpendicular to the chord.

10–8 If a line is perpendicular to a radius at a point on the circle, then the line is tangent to the circle.

10–9 If a line is tangent to a circle, then the radius drawn to the point of contact is perpendicular to the tangent.

10–10 If a line is perpendicular to a tangent at a point on the circle, then the line contains the center of the circle.

10–11 The tangent segments to a circle from a point outside the circle are congruent and form congruent angles with the line joining the center and the point.

10–12 The measure of an inscribed angle is one half the measure of its intercepted arc.

10–13 An angle inscribed in a semicircle is a right angle.

10–14 An angle formed by two chords intersecting inside a circle has a measure equal to one half the sum of the intercepted arcs.

10–15 The measure of an angle formed by a tangent and a chord drawn to the point of contact is one half the intercepted arc.

10–16 The measure of an angle formed by two intersecting tangents to a circle is one half the difference of the measures of the intercepted arcs.

10–17 The measure of an angle formed by a tangent and a secant or two secants from a point exterior to a circle is one half the difference of the measures of the intercepted arcs.

10–18 If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the length of the tangent segment equals the product of the lengths of the secant segment and its external secant segment.

10–19 If two chords intersect in a circle, then the product of the lengths of the segments of one chord equals the product of the lengths of the second chord.

10–20 If two secant segments are drawn to a circle from an exterior point, then the product of the lengths of one secant segment and its external secant segment equals the product of the lengths of the other secant segment and its external secant segment.

Chapter 10—Review

For exercises 1–4 indicate on your paper whether the following statements are true or false. If a statement is false, draw a counterexample.

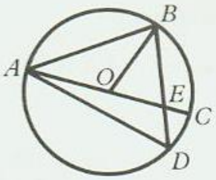
1. If a triangle is inscribed in a circle with one side as a diameter, then the triangle is a right triangle.

2. If a line bisects two chords which are not diameters, then the chords are parallel.

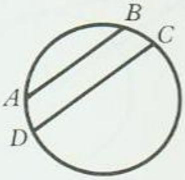
3. If a line is perpendicular to a chord, then it contains the center of the circle.

4. If $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to the same circle at points A and B respectively, then $PA = PB$.

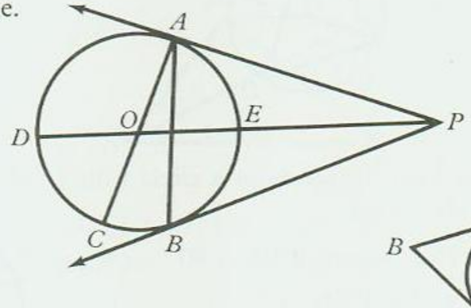
5. **Given:** $m\angle AOB = 120$, $m\widehat{AD} = 150$, $\overline{AC}$ is a diameter.
Find: **a.** $m\angle ADB$
b. $m\angle BAC$
c. $m\angle CED$
d. $m\widehat{BCD}$



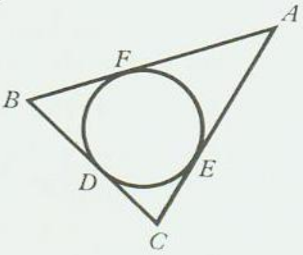
6. **Given:** $m\widehat{AD} = m\widehat{BC}$
Prove: $\overline{AB} \parallel \overline{CD}$



7. **Given:** $\overline{PA}$ and $\overline{PB}$ are tangents.
O is the center of a circle.
$\overline{AC}$ is a diameter.
$m\angle APO = 20$
Find: **a.** $m\angle DPB$
b. $m\widehat{AD}$
c. $m\widehat{BC}$
d. $m\widehat{AE}$
e. $m\angle CAB$

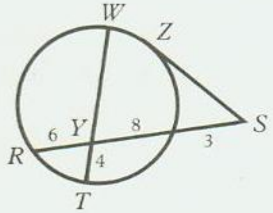


8. **Given:** $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are tangents at points F, D, and E respectively. $CE = 3$, $AF = 5$, $BC = 7$
Find: **a.** length of $\overline{BD}$.
b. perimeter of $\triangle ABC$.



9. If a 16 cm chord is 15 cm from the center, what is the radius of the circle?

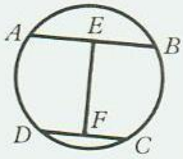
10. $\overline{SZ}$ is tangent to the circle in the figure shown. Find SZ and WY.



Chapter 10—Test

For exercises 1–4 indicate on your paper whether the following statements are true or false. If a statement is false, draw a counterexample.

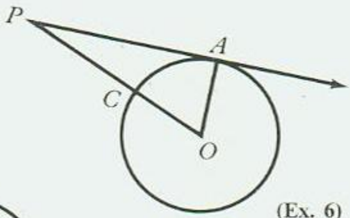
1. In two different circles, if two chords are the same length, they are the same distance from their centers.
2. If a triangle is inscribed in a circle and the intercepted arcs have measures of 200°, 90°, and 70°, then triangle is an obtuse triangle.
3. If a line bisects the minor arc of a chord, it bisects the chord's major arc also.
4. If two tangents are parallel, then their points of tangency determine a diameter.



(Ex. 5)

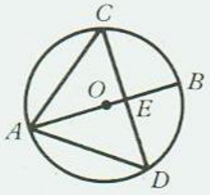
5. **Given:** $\overline{AB} \parallel \overline{CD}$, $\overline{EF}$ is the perpendicular bisector of $\overline{AB}$.
Prove: $\overline{EF}$ bisects $\overline{CD}$.

6. **Given:** $\overline{PA}$ is tangent to circle O.
$OA = 10$ cm, $PA = 24$ cm
Find the length of $\overline{PC}$.



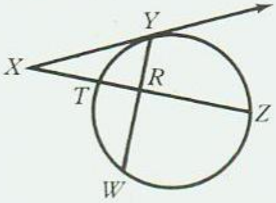
(Ex. 6)

7. **Given:** $\overline{AB}$ is a diameter.
$m\angle CAB = 50$
$m\widehat{BD} = 20$
Find: **a.** $m\widehat{BC}$
b. $m\widehat{AC}$
c. $m\angle ADC$
d. $m\angle BED$

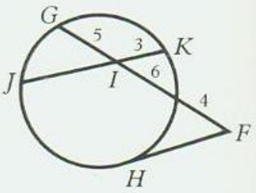


8. Suppose a chord is 6 mm from the center of a circle with a radius of 10 mm. How long is the chord?

9. **Given:** $m\angle X = 30$, $m\widehat{YT} = 50$, $m\angle WRZ = 90$
$\overrightarrow{XY}$ is a tangent to the circle.
Find $m\widehat{YZ}$, $m\widehat{WZ}$, and $m\widehat{TW}$.



10. $\overline{FH}$ is tangent to the circle in the figure shown. Find JI and FH.



Cumulative Review—Chapters 8–10

1. Indicate on your paper whether the following statements are true or false.

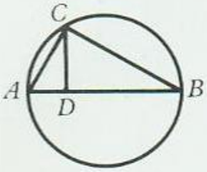
   a. If a parallelogram is inscribed in a circle, it is a rectangle.

   b. If *ABCD* is a parallelogram, then its diagonals are congruent.

   c. If $m\angle A = 40$ in right triangle *ABC* and $m\angle D = 50$ in right triangle *DEF*, then the triangles are similar.

   d. All rectangles are similar polygons.

2. $\overline{AB}$ is a diameter and $\overline{CD}$ is perpendicular to $\overline{AB}$.



   a. If $AD = 4$ and $AB = 12$, find *CD*.

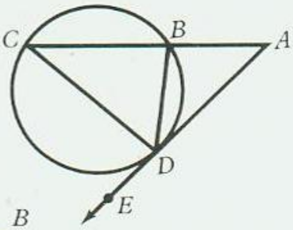
   b. If $AB = 13$ and $CD = 6$, find *AD*.

   c. If $CB = 12$ and $AB = 13$, find *BD*.

   d. Prove $\triangle ABC \sim \triangle CBD$.

3. What is the measure of each interior angle of a regular 10-sided polygon?

4. $\overline{AC}$ is a secant, $\overline{AD}$ is a tangent, $m\widehat{BC} = 100$, $m\angle CBD = 80$, and $m\widehat{BD} = 100$.



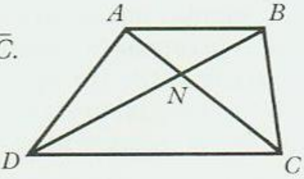
   a. Find $m\angle ABD$.

   b. Find $m\angle BCD$.

   c. Find $m\angle CAD$.

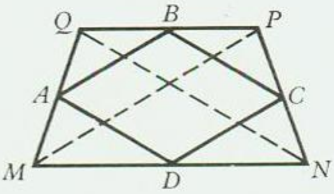
   d. Find $m\angle EDC$.

5. **Given:** *ABCD* is a trapezoid with $\overline{AB} \parallel \overline{DC}$.
   Prove: $NB \cdot NC = NA \cdot ND$



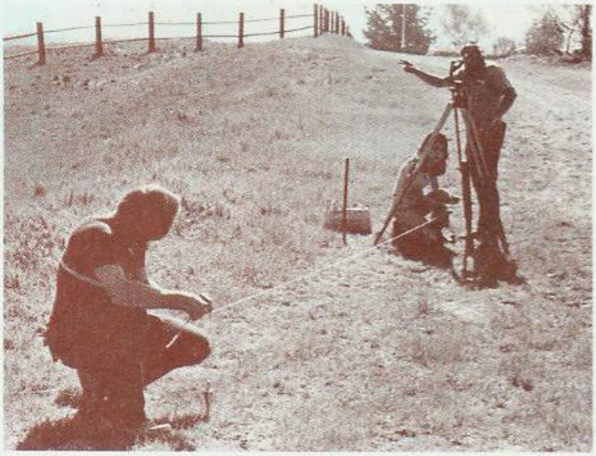
6. If four angles of a pentagon have measures of 100°, 160°, 90°, and 150°, what is the measure of the fifth angle?

7. **Given:** *MNPQ* is a trapezoid with $MP = QN$.
   A, *B*, *C*, and *D* are midpoints as shown.
   Prove: *ABCD* is a rhombus

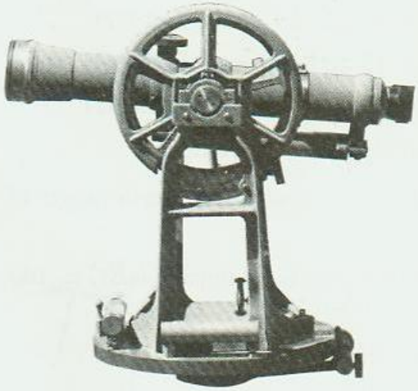


Geometry In Our World

Surveying: The Transit

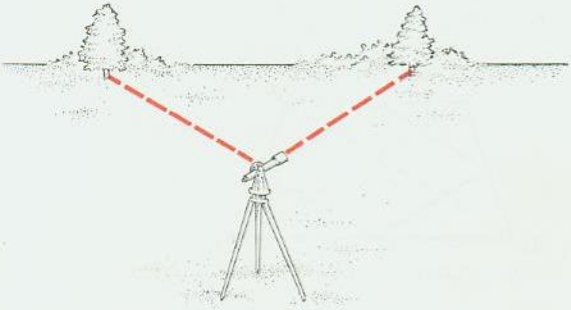


A surveyor is involved in the construction of buildings, roads, dams, and bridges. One of the surveyor's responsibilities is to establish exact land boundaries. All the necessary information is recorded at the site. Later, at the office, drawings or maps are made of the surveyed land.

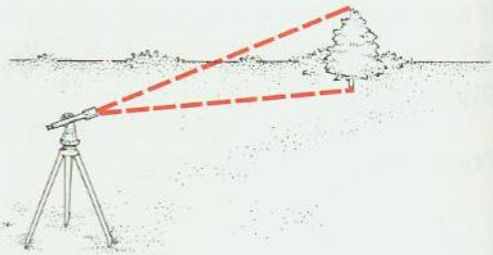


The transit is probably the surveyor's most valuable tool. It is used to measure both angles and distances.

The transit can be used both to measure angles between objects and angles of elevation. An angle between objects is measured by sighting the first object and then moving the telescope to the right or left to the second object. An angle of elevation is measured by tilting the telescope up to sight the top of the object. There are two scales on the transit, one to measure horizontal angles and one to measure vertical angles.

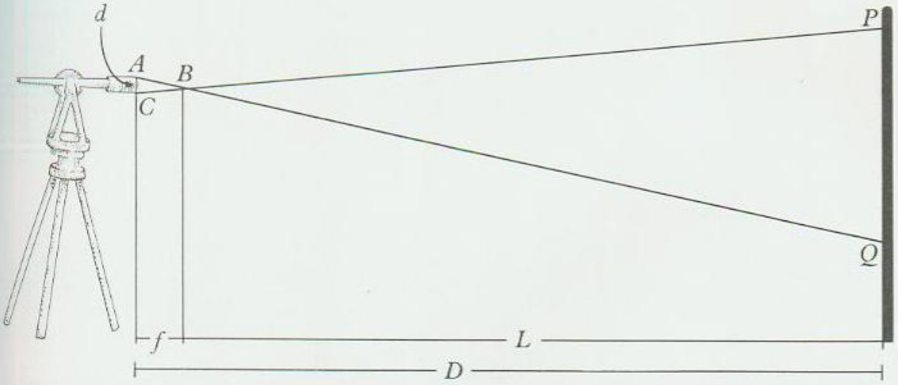


Measuring angle between two objects (horizontal angle)



Measuring angle of elevation (vertical angle)

The transit is also used to measure distances. The optical properties of a transit telescope cause light rays to cross to form a pair of similar triangles.



Two people are usually involved in finding distances. One person stands at the transit, and the other stands at the second location, holding a scaled rod perpendicular to the ground. The following steps show how to find the desired distance D.

Step 1 Look through the telescope. Read the numbers on the rod to determine the distance PQ.

Step 2 Find L by using a proportion based on similar triangles.

Step 3 Add f to L to find D.

1. Why is $\triangle ABC$ similar to $\triangle QBP$?
2. f and L are lengths of the altitudes of the two triangles. We can prove that

$$\frac{f}{L} = \frac{d}{PQ}.$$

Use this proportion to find an expression for $\frac{f}{d}$.

3. f is called the focal distance of the telescope. The telescope is constructed so that the light rays will always cross at the same distance. d is also fixed for a given telescope. Therefore the ratio f/d is fixed.
   Suppose $f = 1$ meter, $d = 1$ centimeter, and $PQ = 0.836$ meters. Find L and D (see Step 3 above).